Rigor in Analysis: From Newton to Cauchy

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Introduction

The invention of the calculus was undoubtedly one of the most important events in the history of mathematics. However, the calculus of Newton and Leibniz, the independent creators of the subject, is much different from the calculus of today. The calculus of today is a calculus based on functions, while the calculus of Newton and Leibniz is a calculus based on geometry. Also, contemporary calculus is much more logically rigorous than at its time of invention. It is this level of rigor in the calculus that will be explored. In the first century and a half of the calculus, from the time of Newton and Leibniz through Cauchy in the 1820s, the level of rigor in the calculus was changing. From the attempts of Newton to be rigorous, to Leibniz and his general indifference to rigor, to the fairly unrigorous work of eighteenth century mathematicians, to the rigor standard setter Augustin-Louis Cauchy, levels of rigor in the calculus during this period were anything but static. Below is a discussion of the differing levels of rigor in the calculus from Newton and Leibniz to Cauchy, followed by a brief analysis of Cauchy's work in making the calculus rigorous. Forces that impeded or encouraged the level of rigor in the calculus will be examined and contributions of individuals who worked with calculus and rigor will be noted.

Newton

While Newton never published a work which solely focused on his version of the calculus, chunks of his work with this discipline appeared in "correspondence, the circulation of manuscripts, and a few published pieces" (Calinger, 607). Pieces of Newton's calculus appeared in works published in 1693, 1707, and 1711. However, according to Newton, he developed his calculus based on the concept of fluxions around 1665-66. A possible explanation for this delay is offered below by Boyer. Nonetheless, using these new methods, Newton did things like find the quadrature of a curve and develop a generalized binomial expansion.

As Calinger writes, "Essential to a mature calculus is a satisfactory foundation" (611). From the beginning, Newton was exploring elements related to rigor (611). Newton's first conception of the calculus used the concept of infinitesimals, values that are greater than zero but less than any quantity, as a basis for his method of determining tangents. However, the use of this concept, although producing generally correct results, is rigorously troubling. When differentiating, ignoring infinitesimal terms led Newton to recognizably correct solutions (Kitcher, 475). However, logical inconsistencies cropped up when one tried to define just how "small" infinitesimals were. If they were non-zero, then equations that had been developed about the derivative would be false. If the were zero, then the equations that had been developed were "illformed" (477). Recognizing this, Newton began to look for another basis on which to set his calculus.

Newton turned his focus to what he called the "prime and ultimate ratio" or "first and last ratio", which was a ratio of infinitesimals that resulted from the formula used to find the slope of a tangent line (Calinger, 614). The formula used was essentially the same used today. While this basis for the calculus was much more rigorous than infinitesimals, Newton left some ambiguity as to what exactly he meant by these ratios. Since Newton's conception of the calculus was in more geometric terms, his view of a limit "was bound up with geometric intuitions which led him to make vague and ambiguous statements" (Boyer, 197). This ambiguity in Newton's work would lead to much debate between his successors about what he actually meant.

Overall, Newton seemed to be concerned with not only using the calculus to solve problems, but also with making sure his conception of the calculus was logically sound. Both Newton's actions and words lend credence to this idea. Newton began to move further away from the use of infinitesimals in his calculus, eventually totally disavowing their use. Also, we note that although Newton completed much of his work with the calculus in the years 1665 to 1676 none of it was published in that period. Boyer comments that "it has been suggested that Newton's long delay in publication of his…works on the calculus was occasioned by the fact that he was dissatisfied with the logical foundations of the subject" (202). Finally, in his 1704 publication, Newton himself noted that "errors are not to be disregarded in mathematics, no matter how small" (201). Thus, despite the leftover ambiguity, Newton did make a conscious effort to make his work rigorous in nature.

Leibniz

Leibniz invented the basic elements of his calculus in October and November of 1675. Like Newton's first conception of the calculus, Leibniz based his calculus on infinitesimals. Unlike Newton, however, Leibniz sought the "algebraization of infinitesimals", rather than using a purely geometric basis (Calinger, 622). Out of this early work came notation that is still used in modern calculus. For what he called, at the suggestion of the Bernoulli brothers, the "integral" (Boyer, 205), Leibniz developed the familiar $\int x \, dx$ notation. For the "differences" (205) in the values of x, Leibniz wrote dx. Leibniz, whom Calinger calls a "masterful notation builder" (623), worked to perfect his symbols. Along with this, Leibniz established the rules for the derivatives of products, quotients, and powers as well as defining the integral as a sum of an infinite number of infinitely narrow rectangles.

Though Leibniz provided much useful notation and many new results to the calculus, compared to Newton his work lacks in rigor. This is due to the fact that Leibniz never stopped using infinitesimals. His refusal to stop using infinitesimals can be attributed to his attitude toward the calculus. Unlike Newton, who believed he was only extending known methods to produce new results, Leibniz realized that he was creating a new discipline (Boyer, 208). Leibniz believed that the new algebra based calculus he was developing had "outstripped the methods of traditional mathematics" (Kitcher, 479). Algebra, he thought, had a wider scope than the mathematics of the time. Thus, he rejected the idea that algebraic reasonings are "only shortcuts for the solution of arithmetical or geometrical problems" (479). Leibniz made it his practice to promote his beliefs by trying to extend algebraic techniques first and worrying about foundations and logical inconsistencies caused by infinitesimals later (479). More concerned about his algorithmic method, which was providing known and correct results, Leibniz struck back at those who assailed him for not being rigorous enough by labeling them "over-precise" critics, likening them to the ancient skeptics (Boyer, 214).

Introduction to Eighteenth Century Mathematics

Before looking at the calculus of the eighteenth century, it is helpful to look at the state in which Newton and Leibniz left the calculus. Both gave to the calculus powerful methods that could immediately be used to solve problems and uncover new results. However, both, but Leibniz more so than Newton, left foundational issues unresolved. Thus, we should not be too surprised to find "confusion among their followers as to the nature of the subject" (Boyer, 221). Although early attempts were made to rigorize the calculus, significant headway was not made until nearly the end of the century. We also see geographical divergence between the work of Leibniz's successors on the European continent and Newton's successors in the British Isles.

The Continental mathematicians paid much more attention to problem solving and less attention to rigor, in the same vein as the German Leibniz, while mathematicians in the British Isles took a much harder look at foundations like their countryman Newton.

Rigor aside, the eighteenth century proved to be such an innovative and important period in the historical development of the sciences that it is often referred to as the 'Scientific Revolution' (Grabiner [1974], 356). New mathematical results were coming at a fast and furious pace. Galda notes that this period has sometimes been referred to as the 'heroic age' of mathematics. He, however, terms it the 'age of vigor', instead. He justifies his coinage of this term by writing, "Although a few mathematicians were worried about the foundations…most plunged ahead…vigorously adding more and more floors to the edifices of mathematics" (135).

Impediments to the Establishment of a Rigorous Calculus

Whatever one prefers to call this epoch in mathematics, it cannot be disputed that many important results, most concerning physical phenomena, were developed during this period. Most often, these results came from the use of the calculus. As Grabiner [1974] writes, "The calculus was an ideal instrument for deriving new results" (356). This philosophy, in which results were highly sought, sometimes without care for rigor, led to an environment about which Grabiner [1974] writes, "The ends justified the means" (356). As long as a result was reached, little attention was paid to the level of rigor used in its derivation. In fact, lack of rigor was sometimes a source of pride for the periods' mathematicians. Galda notes that Lacroix, in the preface of one of his textbooks, boasted "such subtleties as the Greeks worried about we no longer need" (135). The time that would have been necessary to devise proofs of the results that had been shown to work was seen as time that would be better spent developing new results.

Other impediments to the move to add rigor to mathematics existed. One of these impediments was the great reliance of mathematicians on symbols. As Grabiner [1974] writes, "Sometimes it seems to have been assumed that if one could just write down something which was symbolically coherent, the truth of the statement was guaranteed" (356). However, when one is lacking rigorously defined concepts, as was the case with the calculus, the symbolism used to represent these concepts may be faulty in nature. This deep trust in symbolism came from the successes of algebra. Symbolism had helped reveal and explain relations that before were difficult to make sense of (Grabiner [1974], 356). Thus, this often successful use of symbols proved to dissuade mathematicians from prodding deeper into the rigor that lay beneath the manipulation of these notations. Since the symbols seemed to provide correct answers, it did not seem there was any reason to challenge the validity of their usage in this manner.

Finally, a very practical impediment was that placing the calculus on a rigorous footing was a very difficult task, surely not something that could be done in a short time. As Grabiner [1981] writes,

> To make a subject rigorous requires more than just choosing the appropriate definitions for the basic concepts; it is necessary also to be able to prove theorems about these concepts. Developing the methods needed for these proofs is seldom merely a trivial consequence of choosing the right definitions; in fact, the prior existence of the methods of proof is often necessary in order to recognize suitable definitions. A great deal of labor was required to devise the techniques and concepts needed to establish a firm foundation for the diverse results and applications of the calculus (16).

It is not an easy task to quickly and effectively add rigor to subject when one has not been in the habit of making sure the majority of their work meets rigorous standards. Couple this with the lack of a generally accepted method with which one can add rigor and there is an atmosphere in which rigorous proofs of results are not likely to have a very important role.

Compounding the difficulties expounded upon above was the fact that there was a major shift in mathematics occurring in the eighteenth century. While it has been noted that basing the calculus on the idea of limits is the rigorous choice, there was much confusion about how to integrate this concept into a rigorous formation of the calculus. This confusion arose because there was “a lack of a clear distinction between questions of geometry and those of arithmetic, and…the absence of the formal idea of a function” (Boyer, 236). However, on the Continent more so than in Britain, “there was a growing tendency to link the calculus with the formal concept of a function, instead of with the intuitional conceptions of geometry” (236). Uncertainty about what the basis of the calculus would be prolonged the process of developing a rigorous footing for the calculus.

Forces Promoting Rigor in the Calculus

What, then, provided the impetus for the shift in the calculus from a results focused science to a discipline based on the ideas of proofs and rigor? It is hard to pinpoint one specific reason why this shift occurred. However, there are many contributing factors. One of these was that the calculus was made rigorous to avoid and correct errors. While this is a very sensible and intuitive argument, it is actually less applicable than it may seem at first glance. In fact, many eighteenth-century results were accurate, even though they had little rigorous background to prove they were correct. The reasons for this unproven accuracy according to Grabiner [1974] are as follows,

> First, some results could be verified numerically, or even experimentally; thus their validity could be checked without a rigorous basis. Second, and even more important, eighteenth-century mathematicians had an almost unerring intuition. Though they were not guided by rigorous definitions, they nevertheless had a deep understanding of the properties of the basic concepts of analysis. This conclusion is supported by the fact that many apparently shaky eighteenth-century arguments can be salvaged, and made rigorous by properly specifying hypothesis (358).

This is not to say, however, that all results found in the eighteenth-century would later be found correct once calculus concepts were placed on a rigorous basis. There were results that were found to be incorrect (such as the sum of the series 1-1+1-1+1-1…. does not equal 1, -1, .5, or any value for that matter), and placing calculus concepts on a rigorous footing allowed for the necessary corrections to be made. Work in the areas of multivariable functions, complex functions, and trigonometric series (fields in which the truth of plausible conjectures is hard to ascertain), may have helped draw attention to the need to place results on a rigorous foundation (Grabiner [1974], 358).

Another possible motivation for turning to rigor might have been the desire to generalize the multitude of results that had been found. Many results had been discovered, but they were fairly loosely connected in nature. Rigor could serve to unify, prove, and generalize results (Grabiner [1974], 358). One may plausibly conjecture that a time of reflection and organization could be expected to follow a period of exploration. As Kleiner argues, this shift from vigorous exploration to rigorous organization could be seen as a "natural" process (298). He explains: "After close to 200 years of vigorous growth with little thought given to foundations, such foundations as did exist were ripe for reevaluation and reformulation" (298).

A factor that should be mentioned as a possible explanation for a shift to increased rigor in mathematics is the Greek tradition of rigor in geometry. As Grabiner [1974] writes, "Everybody from the Greeks on knew that mathematics was supposed to be rigorous" (359). She also notes [1981] that Cauchy stated that the Greek methods were his model of rigor (30). If this was not enough to get mathematicians to more seriously consider the issue of rigor, George Berkeley, Bishop of Cloyne, attacked the calculus in 1734, "on the perfectly valid grounds that it was not rigorous the way mathematics was supposed to be" (Grabiner [1974], 359). Berkeley charged, in *The Analyst, or a Discourse Addressed to an Infidel Mathematician*, although many of the results found by using the calculus were correct, the methods by which they were obtained were fundamentally unsound (Grabiner [1981], 27). He mocked Newton's so called 'vanishing increments' by writing, "And what are these …vanishing increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?" (Grabiner [1974], 359). While neither the Greek tradition of rigor nor the attack by Bishop Berkeley alone is likely to have caused the shift to more rigor in mathematics, both got mathematicians thinking about rigor. Berkeley's criticisms were especially taken to heart by Lagrange whose influence on the shift to a more rigorous footing is discussed below.

Another influential factor in the move towards rigor in analysis was the move of mathematicians from their positions of attachment to royal courts to teaching positions at newly emerging institutions of higher education. As Kleiner writes, "Most mathematicians since the French Revolution earned their livelihood by teaching" (297). In this role, mathematicians were apt to look much more closely at the foundations of their work (Grabiner [1974], 360). Kleiner discusses this shift to an increased interest in foundations: "Mathematicians presumably think through the fundamental concepts of the subject they are teaching much more carefully when writing for students than when writing for other colleagues" (298). Grabiner [1974] relates this idea to a classroom situation when she writes,

> Teaching always makes the teacher think carefully about the basis for a subject. A mathematician could understand enough about a concept to use it, and could rely on the insight he had gained through his experience. But this does not work with freshmen, even in the eighteenth-century. Beginners will not accept being told, "After you have worked with this concept for three years, you'll understand it" (360).

Grabiner [1974] notes that before the establishment of scientific journals, most work on logical foundations appeared in texts and courses of lectures. Also, even after analysis did take to the forefront in mathematics, most new breakthroughs originated in course lectures (360). These ideas and facts greatly contribute to the argument that the move of mathematicians into the classroom greatly promoted a move to more rigor in the calculus.

Finally, if we were to give one eighteenth century mathematician credit for helping move the calculus to a more rigorous footing, that credit would unquestionably go to Lagrange. As Grabiner [1981] writes, Lagrange was "the first major mathematician to treat the foundations of the calculus as a serious mathematical problem" (37). Lagrange's repeated inquiries into the rigorous foundations of the calculus had much to do with his belief in the validity of the criticisms of Berkeley (37). In fact, Lagrange also "displayed a skeptical attitude toward the infinitely small, echoing Berkeley" (Boyer, 251). Because of this, Lagrange advocated a movement towards rigor in many ways, two of which will be mentioned below: the proposal of a prize problem to the Berlin Academy concerning placing the calculus on a rigorous footing, and his publishing of an influential work on the subject in the late 1700s (Grabiner [1981], 37).

In 1784, at the suggestion of Lagrange, the Berlin Academy proposed a prize problem of constructing the rigorous foundations of the calculus. While this was a valiant attempt at soliciting a solution, the material submitted to the contest did not fit what the Academy wanted. The unrealized goal was to find "a clear and precise theory of what is called *Infinity* in mathematics" (Grabiner [1981], 41). Lagrange and the other judges held the submissions to an extremely high standard and found themselves disappointed with the work submitted. Instead of placing the calculus on a rigorous footing, most of the contributions had tried to find some ad hoc explanatory principle such as compensation of errors. Also, the works had not justified, by rigorous proof, the wealth of known results. The winning submission, made by Simon l'Huilier, was, as Grabiner [1981] writes, "the best of a bad lot" (42).

Despite few worthwhile results from the Berlin prize offering, the competition put on the table the idea that the calculus needed an infusion of rigor to advance and thrive. Lagrange took it upon himself to work on the problem he proposed. In 1797, Lagrange published *Fonctions analytiques*, in which he claimed to have solved the problem of placing the calculus on a rigorous basis. Lagrange's book differed from earlier attempts in that he did more than just give definitions to basic concepts. He used these definitions to derive proofs of the major known results (Grabiner [1981], 37). Although his definition of $f'(x)$ as the coefficient of h in the Taylor series expansion of $f(x+h)$ is not accepted by modern mathematicians, his work was of great importance (Grabiner [1974], 361). Among other things, this work entrenched rigor as an important area of mathematics, developed new techniques to carry out rigorous analysis, and set an example of how to thoroughly prove known results of the calculus from definitions (Grabiner [1981], 37).

Infinite Series Issues

As evidenced by the work of Lagrange and similar methodologies put forth by Condorcet, Arbogast, and Servois among others, we see that the issue of whether to base the calculus on functions or on geometry had firmly been settled in favor of functions by the end of the century (Boyer, 263). More attention could now be turned to the rigorization of the calculus based on the function concept. Lagrange's work also demonstrated another motivation for a move towards stronger foundations, namely, the representation of functions as series. In the opinion of Kitcher [1981], it is this work with infinite series that was "the impetus for making the calculus rigorous" (488).

However, Lagrange was not the only mathematician to use infinite series in their work. Fourier, in 1807, presented to the Institut de France a work in which he claimed that any even function could be made into a "possibly infinite sum of cosines, what we today call a Fourier series" (Bressoud, 3). This result, which came out of Fourier's investigations in to the flow of heat in a very long and thin rectangular plate or lamina, proved to be very controversial as it "contradicted the established wisdom about the nature of functions" (3). Fourier's work raised issues that deeply challenged how the concepts of the calculus had been formulated by asserting that the shape of a function between two values did not determine the behavior of the function over the whole domain (56). The infinite series expansion of a function by Fourier and Lagrange, along with the work of Euler covered below, led to work by Cauchy to add rigor to the calculus in hopes of finding out how much validity existed in these ideas.

Before Fourier and Lagrange, Euler had worked with infinite series in the mid-eighteenth century. Often, using methods Kline describes as "ad hoc", Euler discovered many important results concerning the summation of infinite series. Despite using "efforts to justify [his] work, which, we can now appraise with the advantage of hindsight, often border on the incredible" (307), Euler had an "uncanny" (307) ability to choose methods that led to correct results. In his work, Euler often used things like divergent series, which without an agreed upon method of usage prove to be unrigorous, with few qualms. Even though the use of these series provided him with logical inconsistencies in many situations, he often ignored these and instead focused on when the use of divergent series provided him with correct results (Kitcher, 488).

As noted above, the fundamental difference between the work of Lagrange and Fourier dealt with how a function behaved in a defined interval as compared to how it behaved over its entire domain. For Euler, "the shape of a function between $x=0$ and $x=l$ determined that function everywhere" (Bressoud, 56). Lagrange built on this idea; his Taylor series representation "implie[d] that the values of a function and all of its derivatives at one point completely determine the function at every value of *x*" (56). Fourier, however, argued that the behavior of a function at one point or in an interval did not necessarily determine how the function behaved elsewhere. At first, Lagrange thought there must surely have been an error in Fourier's work; possibly that the cosine functions did not converge like Fourier claimed they did. However, that issue was resolved in Fourier's favor and the larger questions of the rigorous way to define a function, an infinite series, and a derivative became more important than ever (57).

Despite the fact that Lagrange and Fourier had tried to infuse their work with rigor and that Euler came up with many correct results, Cauchy did not feel comfortable with the contradictory results and the level of rigor provided by these mathematicians. However, without an accepted method of rigorization to measure these results against, there was little Cauchy could do to validate his lack of comfort. Kitcher explains the situation well when he writes: "Cauchy's research into the representation of functions by infinite series was crippled by the lack of a reliable procedure for using infinite series expressions" (489). Thus, Cauchy took to the task of adding rigor to the calculus in an attempt to tackle the infinite series questions and other general questions of rigor in analysis.

Cauchy

While Augustin-Louis Cauchy published a prodigious amount of work in a variety of disciplines, it was his work in adding rigor to the calculus that stands out as one of his most important accomplishments. In three works published in the 1820s, *Cours d'analyse* (1821), *Resume des lecons sur le calcul infinitesimal* (1823), and *Lecons sur le calcul differential* (1829), Cauchy moved the bar of rigor to an unprecedented height. He rigorized the notions of such fundamental topics as limits, continuity, convergence, the derivative, and the integral. The result of this, Boyer writes, is that "Cauchy did more than anyone else to impress upon the subject the character which it bears at the present time" (271).

The success Cauchy enjoyed in completing the shift to a more rigorous calculus, according to Grabiner [1981], is due to Cauchy's simultaneous realization of two central concepts. The first was that "the eighteenth-century limit concept could be understood in terms of inequalities" (77). The second, and according to Grabiner, more important, was that prior realization set the stage for "all of the calculus [to] be based on limits" (77). These realizations transformed previous results on such things as continuous functions, infinite series, derivatives, and integrals into modified theorems in this new rigorous analysis.

In choosing the method of basing the calculus on the algebra of inequalities and on limits, Cauchy departed from the previously mentioned work of Lagrange and Fourier. As Katz writes Cauchy did not like the use of infinite series promoted by both Lagrange and Fourier: "Cauchy in fact was not satisfied with what he believed were unfounded manipulations of algebraic expressions, especially infinitely long ones. Equations for these expressions were only true for certain values, those values for which the infinite series was convergent" (707). Using this calculus of limits promoted by such eighteenth century mathematicians as Jean d'Alembert, Cauchy developed a method of adding rigor whose influence has continued to this day.

In looking at how Cauchy added rigor to the concepts of the limit, convergence, continuity, the derivative, and the integral, we see a general methodology emerge. First, Cauchy would take to the task of developing a definition that was rigorous in nature. In doing this, much attention was paid to detail. For example with his definition of a limit given in the *Cours*,

> When the successively attributed values of one variable indefinitely approach a fixed value, finishing by differing from that fixed value by as little as desired, that fixed value is called the *limit* of all the others (Grabiner [1978], 382).

Cauchy appealed to “the notions of number, variable, and function, rather than to the intuitions of geometry and dynamics” (Boyer, 273). These definitions, founded on the more rigorous footings, served as the basis for the rest of Cauchy’s work with the calculus.

Definitions in hand, Cauchy proceeded to fashion theorems for the calculus. However, he did not stop at the development of these theorems; Cauchy devised rigorous proofs for each theorem. Key to these proofs were his newly formulated definitions. Major theorems that Cauchy presented and proved using his rigorous definitions include the Fundamental Theorem of Calculus (Katz, 719) and the Mean Value Theorem for Derivatives (716). Especially in the cases of derivatives and integrals, it was these theorems that were extremely important for the advancement of the calculus. As Grabiner [1978] writes, “There is more to the rigorous theory of the derivative than a mere definition. The mathematical value of Cauchy’s definition stems not from its logical correctness alone, but from the proofs of the theorems to which Cauchy applied it” (389). Thus, the method of providing rigorous definitions and using them in rigorous proofs of theorems proved to be a very fruitful method when used by Cauchy to add rigor to the calculus.

In looking at what Cauchy’s work meant to the calculus, Grabiner [1981] writes, “Cauchy’s work established a new way of looking at the concepts of calculus. As a result, the subject was transformed from a collection of powerful methods and useful results into a mathematical discipline based on clear definitions and rigorous proofs” (164). By providing and insisting on rigor in the calculus, Cauchy’s work began to shift mathematics into a much more rigorous discipline, one more reminiscent of the mathematics of the Greeks than the mathematics of the eighteenth century.

Conclusion

Looking at the level of rigor in the calculus from Newton and Leibniz through Cauchy, we see a period in which the level of rigor ebbed and flowed a considerable amount. From the rigorously minded Newton to the inventive Euler to the standard setting Cauchy, we see three different mathematicians of the era holding themselves to three different rigorous standards. Though the rigorous standards of each differed quite considerably, it is undeniable that each of them made indelible contributions to the calculus. This flow of rigor provided the calculus with a unique and varied set of results that established it as the premier mathematical subject. Thus, while this paper examines the level of rigor in the calculus, it does not aim to condemn the less rigorous work of many eighteenth century mathematicians. Rather, it seeks to trace the use of rigor in the calculus and explain why it changed as it did. While being rigorous is viewed as paramount in modern mathematics, we see from the development of the calculus that at times using lower levels rigor does not lead to the inevitable decline of mathematics as a logically sound science. Kitcher, echoing this view writes, “Who needs mathematical rigor? Some mathematicians at some times, but by no means all mathematicians at all times” (490). In looking at the historical development of rigor in the calculus, this statement rings quite true.

Works Cited

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After Cauchy, work on making mathematics even more rigorous proceeded in two directions. One of these was led by Karl Theodor Wilhelm Weierstrass (1815-1897), who recognized that there were important functions that did not behave as well as some of those Cauchy studied.Â The revolution in rigor that took place in mathematics in the nineteenth century was a response to the problems that had been encountered in teaching and applying calculus for more than a hundred years. It was essential to broaden the range of objects to which calculus could be applied, but that was scarcely possible if the calculus itself were so little understood.Â The Development of the Foundations of Mathematical Analysis from Euler to Riemann. Cambridge, MA: MIT Press, 1970. When a nineteenth-century mathematician spoke about rigor in analysis, or in any other subject, he had several general things in mind. First, every concept of the su bject had to be explicitly defined in terms of concepts whose nature was held to be already known.Â When Cauchy referred in that work to the rigor ofgeometry as the ideal to which he aspired, he had in mind, not diagrams, but logical structure: the way the works of Euclid and Archimedes were constructed.3. Cauchy explicitly distinguished between heuristics and justification.Â Probably instead they were appealing to the idea of motion, as Newton had done in explaining his calculus. Cauchyâ€"Riemann equations. From Wikipedia, the free encyclopedia. Jump to navigation Jump to search. Conditions required of holomorphic (complex differentiable) functions. A visual depiction of a vector X in a domain being multiplied by a complex number z, then mapped by f, versus being mapped by f then being multiplied by z afterwards. If both of these result in the point ending up in the same place for all X and z, then f satisfies the Cauchy-Riemann condition. Mathematical analysis â†' Complex analysis. Complex analysis.