A cheatsheet for Fourier transform conventions.

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1 A cheatsheet for different Fourier integral notations.

Damn. There's too many different notations for the Fourier transform. Examples are:

$$
\begin{aligned}
\tilde{f}(k) &= \int_{-\infty}^{\infty} f(x) \exp\left(-2\pi i k x\right) dx \\
\tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp\left(-i k x\right) dx \\
\tilde{f}(p) &= \sqrt{\frac{1}{2\pi\hbar}} \int_{-\infty}^{\infty} f(x) \exp\left(\frac{-i p x}{\hbar}\right) dx
\end{aligned}
$$

There's probably many more, with other variations such as using hats over things instead of twiddles, and so forth.

Unfortunately each of these have different numeric factors for the inverse transform. Having just been bitten by rogue factors of 2π after inoccently switching notations, it seems worthwhile to express the Fourier transform with a general fudge factor in the exponential. Then it can be seen at a glance what constants are required in the inverse transform given anybody's particular choice of the transform definition.

Where to put all the factors can actually be seen from the QM formulation since one is free to treat $\hbar$ as an arbitrary constant, but let's do it from scratch in a mechanical fashion without having to think back to QM as a fundamental.

Suppose we define the Fourier transform as

$$
\begin{aligned}
\tilde{f}(s) &= \kappa \int_{-\infty}^{\infty} f(x) \exp\left(-i\alpha s x\right) dx \\
f(x) &= \kappa' \int_{-\infty}^{\infty} \tilde{f}(s) \exp\left(i\alpha x s\right) ds
\end{aligned}
$$

Now, what factor do we need in the inverse transform to make things work out right? With the Rigor Police on holiday, let's expand the inverse transform integral in terms of the original transform and see what these numeric factors must then be to make this work out.

Omitting temporarily the κ factors to be determined we have

$$
\begin{aligned}
f(x) &\propto \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(u) \exp\left(-i\alpha s u\right) du \right) \exp\left(i\alpha x s\right) ds \\
&= \int_{-\infty}^{\infty} f(u) du \int_{-\infty}^{\infty} \exp\left(i\alpha s(x-u)\right) ds \\
&= \int_{-\infty}^{\infty} f(u) du \lim_{R\to\infty} 2\pi \frac{1}{\pi\alpha(x-u)} \sin\left(\alpha R(x-u)\right) \\
&= \int_{-\infty}^{\infty} f(u) du 2\pi\delta\left(\alpha(x-u)\right) \\
&= \frac{1}{\alpha} \int_{-\infty}^{\infty} f(v/\alpha) dv 2\pi\delta\left(\alpha x - v\right) \\
&= \frac{2\pi}{\alpha} f((\alpha x)/\alpha) \\
&= \frac{2\pi}{\alpha} f(x)
\end{aligned}
$$

Note that to get the result above, after switching order of integration, and assuming that we can take the principle value of the integrals, the usual ad-hoc sinc and exponential integral identification of the delta function was made

$$
\begin{aligned}
\text{PV} \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(isx\right) ds &= \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \exp\left(isx\right) ds \\
&= \lim_{R\to\infty} \frac{\sin\left(Rx\right)}{\pi x} \\
&\equiv \delta\left(x\right)
\end{aligned}
$$

The end result is that we will need to fix

$$
\kappa\kappa' = \frac{\alpha}{2\pi}
$$

to have the transform pair produce the desired result. Our transform pair is therefore

$$\tilde{f}(s) = \kappa \int_{-\infty}^{\infty} f(x) \exp\left(-i\alpha s x\right) dx \Leftrightarrow f(x) = \frac{\alpha}{2\pi\kappa} \int_{-\infty}^{\infty} \tilde{f}(s) \exp\left(i\alpha s x\right) ds \quad (1)$$

2 A survey of notations.

From 1 we can express the required numeric factors that accompany all the various forward transforms conventions. Let's do a quick survey of the bookshelf, ignoring differences in the i's and j's, differences in the transform variables, and so forth.

From my old systems and signals course, with the book [Haykin(1994)] we have, $\kappa = 1$, and $\alpha = 2\pi$

$$\begin{aligned}
\tilde{f}(s) &= \int_{-\infty}^{\infty} f(x) \exp\left(-2\pi i s x\right) dx \\
f(x) &= \int_{-\infty}^{\infty} \tilde{f}(s) \exp\left(2\pi i s x\right) ds
\end{aligned}$$

The mathematician's preference, and that of [Bohm(1989)], and [Byron and Fuller(1992)] appears to be the nicely symmetrical version, with $\kappa = 1/\sqrt{2\pi}$, and $\alpha = 1$

$$\begin{aligned}
\tilde{f}(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp\left(-i s x\right) dx \\
f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(s) \exp\left(i s x\right) ds
\end{aligned}$$

From the old circuits course using [Irwin(1993)], and also in the excellent text [Le Page and LePage(1980)], we have $\kappa = 1$, and $\alpha = 1$

$$\begin{aligned}
\tilde{f}(s) &= \int_{-\infty}^{\infty} f(x) \exp\left(-i s x\right) dx \\
f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(s) \exp\left(i s x\right) ds
\end{aligned}$$

and finally, the QM specific version from [McMahon(2005)], with $\alpha = p/\hbar$, and $\kappa = 1/\sqrt{2\pi\hbar}$ we have

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} f(x) \exp\left(-\frac{ipx}{\hbar}\right) dx$$
$$f(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \tilde{f}(p) \exp\left(\frac{ipx}{\hbar}\right) dp$$

References

[Bohm(1989)] D. Bohm. *Quantum Theory*. Courier Dover Publications, 1989.

[Byron and Fuller(1992)] F.W. Byron and R.W. Fuller. *Mathematics of Classical and Quantum Physics*. Dover Publications, 1992.

[Haykin(1994)] S.S. Haykin. Communication systems. 1994.

[Irwin(1993)] J.D. Irwin. *Basic Engineering Circuit Analysis*. MacMillian, 1993.

[Le Page and LePage(1980)] W.R. Le Page and W.R. LePage. *Complex Variables and the Laplace Transform for Engineers*. Courier Dover Publications, 1980.

[McMahon(2005)] D. McMahon. *Quantum Mechanics Demystified*. McGraw-Hill Professional, 2005.

The Fourier transform is an integral transform widely used in physics and engineering. They are widely used in signal analysis and are well-equipped to solve certain partial differential equations. The convergence criteria of the Fourier...Â This article has been viewed 118,565 times. Learn more The Fourier transform is an integral transform widely used in physics and engineering. They are widely used in signal analysis and are well-equipped to solve certain partial differential equations. Fourier analysis is a method for expressing a function as a sum of periodic components, and for recovering the signal from those components. When both the function and its Fourier transform are replaced with discretized counterparts, it is called the discrete Fourier transform (DFT). The DFT has become a mainstay of numerical computing in part because of a very fast algorithm for computing it, called the Fast Fourier Transform (FFT), which was known to Gauss (1805) and was brought to light in its current form by Cooley and Tukey [CT65]. Press et al. [NR07] provide an accessible introduction to The Fourier Transform is one of deepest insights ever made. Unfortunately, the meaning is buried within dense equations: Yikes. Rather than jumping into the symbols, let's experience the key idea firsthand. Here's a plain-English metaphor: What does the Fourier Transform do? Given a smoothie, it finds the recipe. How?Â The Fourier Transform takes a time-based pattern, measures every possible cycle, and returns the overall "cycle recipe" (the amplitude, offset, & rotation speed for every cycle that was found). Time for the equations? No! Let's get our hands dirty and experience how any pattern can be built with cycles, with live simulations. If all goes well, we'll have an aha! moment and intuitively realize why the Fourier Transform is possible. 5.1 Fourier transform from Fourier series. Consider the Fourier series representation for a periodic signal comprised of a rectangular pulse of unit width centered on the origin. In this exposition, however, we donâ€™t specify the period T â€” instead we leave it as a parameter. We denote the signal by xT (t). Some dierent cases are shown below