

## A Critique of Impure Unreason

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Almost 25 years ago, I took the title of a well-known article by Richard W. Hamming (1980), “The Unreasonable Effectiveness of Mathematics,” and turned it around, entitling my article “The Reasonable Ineffectiveness of Research in Mathematics Education” (Kilpatrick, 1981). That inversion allowed me to explain why research in mathematics education invariably seems so ineffective. In the present paper, I have done it again. By negating adjective and noun in the title of Immanuel Kant’s (1781/1998) *Critique of Pure Reason*, I have created a construct—*impure unreason*—that allows me to explore what some are claiming about mathematical reasoning as well as why some are questioning all efforts to reform school mathematics.

I am concerned in this paper with the following question: *In these postmodern times, can and should mathematical reasoning be taught?* Two centuries ago, Kant wanted to show that human autonomy is compatible with the truth of modern science. He defended the claim that a priori knowledge is possible while criticizing and limiting the traditional metaphysical view that rejected empirical scientific knowledge. He said, “There is properly no antithetic of pure reason at all” (Kant, 1781/1988, p. 645). Lately, some have questioned whether students can be taught to reason mathematically. More seriously, some views amount to questioning the status of reasoning in general, which calls mathematical reasoning into question. And some people, questioning efforts to reform school mathematics, engage in what I term impure unreason.

Consideration of these views and efforts leads to a second question: *What do research studies and other scholarly literature in mathematics education have to say about the teaching of mathematical reasoning?*

### *The Mathematics Learning Study*

In 1995, the U.S. Department of Education and the U.S. Department of Human Services asked the National Academy of Sciences to establish a committee to examine the prevention of reading difficulties. Three years later, the committee published its report (Snow, Burns, & Griffin, 1998), which was widely acclaimed as putting an end to

the so-called reading wars. In 1998, noting the positive reception given the reading study and concerned about the so-called math wars, the Department of Education and the National Science Foundation asked the Academy to conduct a similar study on mathematics learning. A Mathematics Learning Study committee was appointed in 1999. Two years later, it published its 480-page report (Kilpatrick, Swafford, & Findell, 2001). Then to disseminate the results to a broader audience, the committee published a 52-page version (Kilpatrick & Swafford, 2002).

The goals of the Mathematics Learning Study were to make recommendations for improving student learning of mathematics, and specifically

- To synthesize the rich and diverse research on pre-kindergarten through eighth-grade mathematics learning.
- To provide research-based recommendations for teaching, teacher education, and curriculum for improving student learning and to identify areas where research is needed.
- To give advice and guidance to educators, researchers, publishers, policy makers, and parents. (Kilpatrick et al., 2001, p. 3)

The Mathematics Learning Study committee comprised 16 people with expertise in various domains: classroom practice, the mathematical sciences, research in mathematics education, research in cognitive psychology, and business. To accomplish its charge, the committee had to do the following: (a) characterize “successful mathematics learning,” (b) identify areas of mathematics that are important foundations in grades pre-K to 8 for building continued learning and then decide how much of that mathematics to consider in the limited time available, (c) discuss the role of research in influencing and informing practice and decide what to count as research evidence, (d) review and synthesize relevant research and policy recommendations about student learning, and finally (e) issue a report with findings and recommendations.

To focus its work and get away from the unproductive dichotomy of skills versus understanding, the committee introduced the construct of *mathematical proficiency*, defining it as composed of five intertwined strands (Figure 1):

- *Conceptual understanding*—comprehension of mathematical concepts, operations, and relations

- *Procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- *Strategic competence*—ability to formulate, represent, and solve mathematical problems
- *Adaptive reasoning*—capacity for logical thought, reflection, explanation, and justification
- *Productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy. (Kilpatrick et. al, 2001, p. 5)

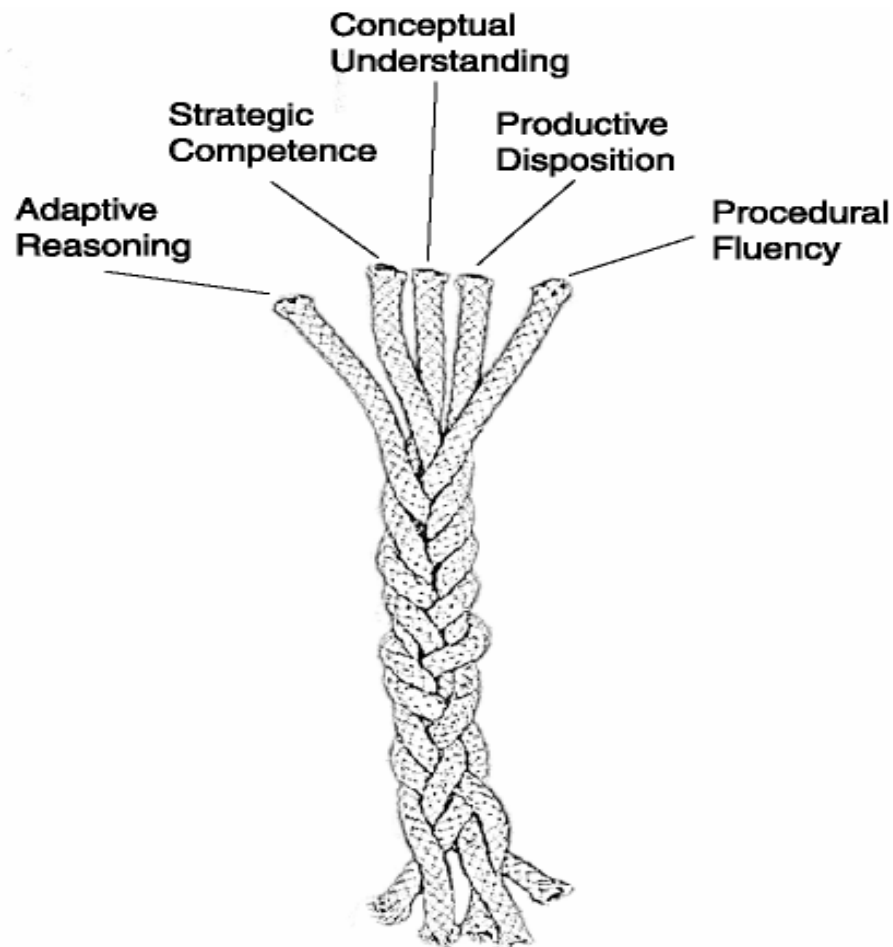


Figure 1. The strands of mathematical proficiency.

Parenthetically, the new Singapore mathematics framework (Singapore Ministry of Education, 2000a, 2000b)—developed completely independently—has five somewhat similar facets organized, like those of mathematical proficiency, around a central core of mathematical problem solving: concepts, skills, metacognition, processes, and attitudes.

### *Adaptive Reasoning*

Although the strand model, the specific strands, and their names took some time for the Mathematics Learning Study committee to formulate and agree on, committee members wanted from the outset to make sure that ideas of reasoning and proof were included in their description of successful mathematics learning. The early effort by the National Council of Teachers of Mathematics (1989) to formulate standards for school mathematics had come under severe criticism for seeming to give insufficient emphasis to proof (Kilpatrick, 1997; Wu, 1997), and the committee wanted to emphasize the central role played in mathematics learning by formulating arguments and providing a justification for them. The first name proposed for the reasoning strand was “logical reasoning,” but that was rejected as being too narrow. The committee eventually settled on “adaptive reasoning.”

*Adaptive reasoning* refers to “the capacity to think logically about the relationships among concepts and situations. Such reasoning is correct and valid, stems from careful consideration of alternatives, and includes knowledge of how to justify conclusions” (Kilpatrick et al., 2001, p. 129). As the report goes on to say, “In mathematics, adaptive reasoning is the glue that holds everything together, the lodestar that guides learning” (p. 129).

Many conceptions of mathematical reasoning are confined to formal proof and other forms of deductive reasoning. Adaptive reasoning is much broader, including not only informal explanation and justification but also intuitive and inductive reasoning based on pattern, analogy, and metaphor. Analogical reasoning, metaphors, and mental and physical representations are “tools to think with,” often serving as sources of hypotheses, sources of problem-solving operations and techniques, and aids to learning and transfer. Many members of the Mathematics Learning Study Committee were guided in their formulation of the adaptive reasoning strand by George Pólya’s views of reasoning.

*Pólya's Views of Reasoning*

In his book *Mathematics and Plausible Reasoning*, Pólya (1954) argues that mathematical reasoning can be divided into *demonstrative reasoning* and *plausible reasoning*. Demonstrative reasoning is used to secure mathematical knowledge; plausible reasoning is used to support the conjectures that give rise to that knowledge. There is a great difference between the two. “Demonstrative reasoning is safe, beyond controversy, and final. Plausible reasoning is hazardous, controversial, and provisional” (vol. 1, p. v).

Pólya goes on to argue that plausible reasoning, despite its important role, has received relatively little attention in discussions of mathematics. He points out that plausible reasoning can be either inductive or analogical. Induction allows us to examine the consequences of a conjecture: “*A conjecture becomes more credible by the verification of any new consequence*” (Polya, 1954, vol. 1, p. 22, italics in original). Analogy leads us to related conjectures: “*A conjecture becomes more credible if an analogous conjecture becomes more credible*” (p. 22, italics in original). Pólya’s book is filled with examples from history of the ways in which mathematicians such as Euler and Archimedes used induction and analogy to support their conjectures. In the final chapter of the second volume, he says to the teacher, “*Let us teach guessing!*” (vol. 2, p. 158, italics in original), arguing that teachers should teach both demonstrative and plausible reasoning.

More valuable than any particular mathematical fact or trick, theorem, or technique is for the student to learn two things:

First, to distinguish a valid demonstration from an invalid attempt, a proof from a guess.

Second, to distinguish a more reasonable guess from a less reasonable guess. (pp. 158–159)

Much of the rest of Pólya’s life was devoted to elaborating this argument, helping teachers of mathematics understand the value of including guesses—and plausible reasoning in general—in their instruction.

*Performance in Reasoning*

How well are students in the United States learning to reason in their mathematics classes? Consider some evidence about the status of students' reasoning from the U.S. National Assessment of Educational Progress. In 1986, 9-year-olds and 13-year-olds were asked this question (Lindquist, 1989, p. 69):

If  $49 + 83 = 132$  is true, which of the following is true?

$$49 = 83 + 132$$

$$49 + 132 = 83$$

$$132 - 49 = 83$$

$$83 - 132 = 49$$

Since all four choices involved the same three numbers, students had only to decide which number sentence was the most plausible. It was disappointing, therefore, that only 61% of the 13-year-olds chose the right answer. That number was many fewer than could find the sum of 49 and 83.

In 1979, 13-year-olds had been given the following problem (cited in Kilpatrick et al., 2001, p. 139):

Estimate  $\frac{12}{13} + \frac{7}{8}$ .

The children had four choices: 1, 2, 19, and 21. Only 24 percent of the 13-year-olds surveyed chose 2 as the estimate; 55 percent chose either 19 or 21. The majority, therefore, were simply manipulating numbers—adding numerators or adding denominators—rather than reasoning about their magnitude.

As a final example, consider the following problem (Wearne & Kouba, 2000, p. 186):

In 1980 the populations of Town A and Town B were 5,000 and 6,000, respectively. The 1990 populations of Town A and Town B were 8,000 and 9,000, respectively.

Brian claims that from 1980 to 1990 the populations of the two towns grew by the same amount. Use mathematics to explain how Brian might have justified his claim.

Darlene claims that from 1980 to 1990 the population of Town A grew more. Use mathematics to explain how Darlene might have justified her claim.

The following table gives the results for 8th graders and 12th graders:

<b>NAEP Results</b>	<b>Grade 8</b>	<b>Grade 12</b>
Correct response for both claims	1%	3%
Partial response	21%	24%
Incorrect response	60%	56%
Omitted	16%	16%

In this item, Darlene's claim is stated in a somewhat cryptic way, and the students may not have understood that they needed to think about population growth multiplicatively rather than additively so as to conclude that Town A had the larger growth rate. But the low levels of performance suggest that the ambiguity of Darlene's claim was not the only problem that the students faced. Few of them, perhaps, had had the opportunity to provide explanations and justifications for a mathematical argument. One of Pólya's most important activities after coming to the United States in 1940 involved a mathematics examination designed to give students that experience.

#### *Examples of Mathematical Reasoning*

When I was a high school student in California in the early 1950s, I took the Stanford University Competitive Mathematics Examination, which was given annually from 1946 to 1965. The examination provided a challenge to my mathematical reasoning skills since I had never before been given three hours in which to solve mathematics problems, let alone being asked to solve only three problems in that amount of time.

The Stanford examination was modeled on the Eötvös Competition in Hungary. Gabor Szegő, chairman of the mathematics department at Stanford in 1946 and winner of the Eötvös competition in 1912, initiated the Stanford examination, but later it was conducted almost entirely by his collaborator and colleague Pólya. The first Stanford examination was administered in 60 California high schools to 322 participants. The winner was awarded a one-year scholarship to Stanford University; honorable mention and a mathematics book were given to three other participants. In 1953, the examination was extended beyond California to include Arizona, Oregon, and Washington; the number of scholarships was increased to two; and the number of honorable mention awards and books was increased to 10 or so. From 1958 to 1962, the examination was co-sponsored by the Sylvania Electric Products Company. The last examination, in 1965, was administered to about 1200 participants in 151 centers in California, Arizona, Idaho, Montana, Nevada, Oregon, and Washington. Cash prizes of \$500, \$250, and \$250 were

awarded to the three winners; honorable mention and a mathematics book went to 18 participants. The examination was discontinued after 1965 mainly because the Stanford Department of Mathematics turned its interest to more graduate teaching.

The problems on the Stanford examination emphasized “originality and insight rather than routine competence” (Polya & Kilpatrick, 1974, p. 1). Although the mathematical content of the problems did not go beyond that of the high school curriculum, the problems were of types seldom found in textbooks. The purpose of such problems was not only to test the students’ reasoning abilities but also to suggest some new directions for the high school mathematics program.

Here is a problem from the 1952 examination (the one that I took):

Prove the proposition: If a side of a triangle is less than the average (arithmetic mean) of the two other sides, the opposite angle is less than the average of the other two angles.

A hint to get you started is to ask, What is the hypothesis? What is the conclusion? Let  $a$ ,  $b$ , and  $c$  denote the sides, and  $A$ ,  $B$ , and  $C$  the opposite angles, respectively. Then use that notation to express the hypothesis and the conclusion. Look at the conclusion and ask whether it might be restated in another way.

Here is another problem, this time from the 1962 examination (one that I helped to mark):

Each of the four numbers  $a$ ,  $b$ ,  $c$ , and  $d$  is positive and less than 1.

Show that not all four products

$$4a(1 - b), 4b(1 - c), 4c(1 - d), 4d(1 - a)$$

are greater than 1.

A hint is to notice that the set of four products is symmetric in  $a$ ,  $b$ ,  $c$ , and  $d$ , so that symmetry ought somehow to be mirrored in the solution.

As you work on the solutions to these problems, think about the reasoning in which you are engaging. What kinds of argument are you using? How are you sure that what you have done is correct?



*Teaching Mathematical Reasoning*

Can the type of reasoning required to solve the problems in the Stanford mathematics examination be taught? Certainly, Pólya believed that it could be. He wrote and spoke extensively about pedagogy during the 1960s and 1970s.

*Pólya's Principles*

In Pólya's (1962 & 1965/1981) book *Mathematical Discovery*, he offers three principles of learning:

*Active learning.* "For efficient learning, the learner should discover by himself [or herself] as large a fraction of the material to be learned as is feasible under the given circumstances." (vol. 2, p. 103)

*Best motivation.* "For efficient learning, the learner should be interested in the material to be learned and find pleasure in the activity of learning. Yet, beside these best motives for learning, there are other motives too, some of them desirable." (vol. 2, p. 103)

*Consecutive phases.* "For efficient learning, an exploratory phase should precede the phase of verbalization and concept formation and, eventually, the material learned should be merged in, and contribute to, the integral mental attitude of the learner." (p. 104)

The consecutive phases principle is based on a sentence from Kant, who is quoted by Pólya (1962 & 1965/1981) as saying, "*Thus all human cognition begins with intuitions, proceeds from thence to conceptions, and ends with ideas*" (quoted in vol. 2, p. 103, italics in original). Pólya then offers his reading of that dictum: "*Learning begins with action and perception, proceeds from thence to words and concepts, and should end in desirable mental habits*" (vol. 2, p. 103, italics in original).

Pólya (1962 & 1965/1981) proceeds to offer three parallel principles of teaching:

*Active learning.* "What the teacher says in the classroom is not unimportant, but what the students think is a thousand times more important. The ideas should be born in the students' mind and the teacher should act only as midwife." (vol. 2, p. 104)

*Best motivation.* "The teacher should pay attention to the choice, the formulation, and a suitable presentation of the problem he [or she] proposes. The problem should appear as meaningful and relevant from the student's standpoint." (vol. 2, p. 105)

*Consecutive phases.* “The trouble with the usual problem material of the high school textbooks is that they contain almost exclusively merely routine examples. Such routine examples may be useful and even necessary, . . . but they miss two important phases of learning: the exploratory phase and the phase of assimilation.” (vol. 2, p. 106)

Pólya (1962 & 1965/1981) contended that these principles should penetrate not only the daily work of the teacher but also the planning of the whole curriculum, the planning of each course in the curriculum, and the planning of each course unit. After having favorably cited Kant as the basis for his ideas on consecutive phases, however, Pólya distances himself from philosophy:

In teaching as in several other things, it does not matter much what your philosophy is or is not. It matters more whether you have a philosophy or not. And it matters very much whether you try to live up to your philosophy or not. The only principles of teaching which I thoroughly dislike are those to which people pay only lip service. (vol. 2, p. 106)

#### *Developing Proficiency in Teaching Reasoning*

Chapter 10 of *Adding It Up* (Kilpatrick et al., 2001) addresses the question of what teachers need to know if they are to teach proficiently and considers some programs to develop proficient teaching. Programs of four types are discussed:

- Focus on mathematics
- Focus on student thinking
- Focus on cases
- Focus on lesson study

Any of these programs can be used to help teachers develop proficiency as described in the book. The approaches can be used well or poorly. To be effective, a program needs to help teachers connect their knowledge of the mathematics they teach, of how their students learn that mathematics, and of how to facilitate that learning. From that point of view, professional development is a form of problem solving.

#### *Teaching Reasoning*

Can students be taught reasoning? Despite Pólya’s claims, empirical research appeared to say otherwise. Beginning in 1900, for example, Edward L. Thorndike and Robert S. Woodworth conducted a series of experiments to show the limitations of

transfer of training unless the tasks had “identical elements.” One of their goals was to test the doctrine of formal discipline, which contended that practice by learning Latin and other difficult subjects, such as geometry, developed the learner’s general skills of learning and attention. Thorndike and Woodworth’s studies raised questions about the possibility of developing one’s “mental muscle.” Instead, people seemed to learn things that were more specific. In particular, Thorndike and Woodworth found that the study of geometry provided no advantage on reasoning tests. Psychologists concluded that advanced mathematics had little power to promote reasoning ability, although teachers of mathematics continued to argue for the value of their subject in helping students learn to think and reason.

More recently, researchers such as Alan Schoenfeld (1985) and Dick Lesh (1981) argued against teaching the “general heuristics” that Pólya advocated. Their claims have helped to lead a generation of researchers in mathematics—at least in the United States—to turn away from the investigation of mathematical problem solving and its promotion. But other forces have been at work as well.

#### *Postmodernism in Mathematics Education*

A number of today’s researchers in mathematics education appear to have been attracted by romantic postmodern views concerning mathematics and its teaching and learning (e.g., Zawadowski, 1990). For example, many claim that all mathematics is situated and relative; consequently, it needs to be dealt with that way in teaching. Teachers are cautioned against interfering with the knowledge construction activities of the learner. Some researchers go even further to argue that knowledge claims about learning and teaching neither can nor need be justified; they are indeterminate texts constructed by readers. Under such an argument, one ends up concluding that all justifications are equal, and nobody’s logic is superior to anyone else’s

#### *Rethinking Research*

Researchers in mathematics education appear to be reacting against this approach. It seems increasingly likely that over the next decade or so, they will be rethinking their research approaches, adopting conceptions of research in which reliable evidence is gathered in a systematic fashion, justifications are offered for arguments, knowledge claims are tested, and the resulting knowledge is made public and verifiable. Our

research will then return to being scientific in the sense of an interpretive science of human thought and behavior.

The so-called math wars in the United States have polarized the community of people concerned professionally with mathematics education, but that conflict appears to be abating somewhat. Regardless of what direction the field takes, research in mathematics education will continue to provide evidence to help us figure out the role that reasoning can play in teaching and learning mathematics.

Research evidence alone—whether synthesized from existing findings or generated by new studies—will change few minds about the direction school mathematics should take. Those calling the loudest for research can be counted on to pay the least attention to it. But research in mathematics education is not about winning debates. It is about providing a foundation of reliable knowledge to improve professional practice. It needs to be deployed not as a weapon but as a tool for construction.  
(Kilpatrick, 2001, p. 426)

### *Conclusion*

I remember my visit to Lisbon in April 1998 on my way to the meeting in Mirandela, where I had been invited to offer one of the reactions to a report prepared by João Pedro da Ponte, José Manuel Matos, and Paulo Abrantes that reviewed a decade or so of mathematics education research and curriculum development in Portugal. Paulo and I spent much of a day together, visiting several locations where he was working on a curriculum project and on professional development activities for teachers. We talked about problem solving, about research on problem solving, and about the difficulty of developing a school mathematics curriculum in which problem solving might play a major role. He was clearly much concerned that students should have experiences and encounter mathematical problems in school that would equip them to reason intelligently about the world of the future. Like Immanuel Kant and George Pólya before him, Paulo Abrantes remained optimistic not only that we can learn to reason inductively and deductively but also that mathematics can play a fundamental role in teaching us how to think.

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Second, both reason and unreason are present in everyone to varying degrees in different ways. Third, Kant invalidates the division as such by characterizing what should be more incomprehensible than an extreme case of unreason as also being a rational way of life. This is a preview of subscription content, log in to check access. Access options. Buy single article. Critique of Pure Reason up to the end of the Analytic. Immanuel Kant 1781. Copyright © Jonathan Bennett 2017. All rights reserved. [Brackets] enclose editorial explanations. Small dots enclose material that has been added, but can be read as though it were part of the original text. I am talking about a critique of the faculty of reason as such, in regard to every attempt it might make to gain knowledge independently of all experience. [In that sentence, "knowledge" translates Erkenntnisse, which is a plural noun. We cannot say "knowledges"; and in contexts where the singular "knowledge" won't do (as it will here), the phrase "items of knowledge", or one of its. 1 One occasionally hears complaints about the superficiality of our age's way of thinking, and about the decline of solid science. Empire of Unreason also avoids the faults of the previous books, which is definitely a good thing. He no longer (or at least, much less noticeably) begins chapters in the middle of the action and has the characters either tell or hear about what happened before. There are far fewer coincidences involved in this book as well with everything having a good reason for happening besides trying to get all of the characters together. Too bad that every time Keyes fixes a flaw, something else creeps in, but it is impossible for a book to be perfect. All in all, Empire of Unreason is yet another winner The Critique of Pure Reason. by Immanuel Kant Trans. J.M.D. Meiklejohn. In form, there are two indispensable conditions, which any one who undertakes so difficult a task as that of a critique of pure reason, is bound to fulfill. These conditions are certitude and clearness. As regards certitude, I have fully convinced myself that, in this sphere of thought, opinion is perfectly inadmissible, and that everything which bears the least semblance of an hypothesis must be excluded, as of no value in such discussions.