Self-presentation

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4. Indication achievements under Art. 16.2 of the Act of 14 March 2003 on Academic Degrees and Title and Degrees and Title in the Arts (Journal of Laws No. 65, item. 595, as amended.): **series of 9 publications**

   a) title of the scientific achievement
   Difference operators on regular lattices admitting Darboux type transformations.

b) The achievement consists of the series of the following 9 publications

[H1] M. Nieszporski, P.M. Santini, and A. Doliwa, 2004, *Darboux transformations for 5-point and 7-point self-adjoint schemes and an integrable discretization of the 2D Schrodinger operator*, Physics Letters A, 323(3–4), 241–250.

[H2] P.M. Santini, M. Nieszporski, and A. Doliwa, 2004, *Integrable generalization of the Toda law to the square lattice*, Physical Review E, 70(5), 056615:1–056615:6.

[H3] M. Nieszporski and P.M. Santini, 2005, *The self-adjoint 5-point and 7-point difference operators, the associated Dirichlet problems, Darboux transformations and Lelieuvre formulae*, Glasgow Mathematical Journal, 47(A), 133–147.

[H4] P. Małkiewicz and M. Nieszporski, 2005, *Darboux transformations for q-discretizations of 2d second order differential equations*, Journal of Nonlinear Mathematical Physics, 12(Supplement: 2), 231–239.

[H5] A. Doliwa, P. Grinevich, M. Nieszporski, and P.M. Santini, 2007, *Integrable lattices and their sublattices: From the discrete Moutard (discrete Cauchy-Riemann) 4-point equation to the self-adjoint 5-point scheme*, Journal of Mathematical Physics, 48(1), 013513:1–013513:28.

[H6] M. Nieszporski, 2007, *Darboux transformations for a 6-point scheme*, Journal of Physics A-Mathematical and Theoretical, 40(15), 4193–4205.

[H7] A. Doliwa, M. Nieszporski, and P. M. Santini, 2007, *Integrable lattices and their sublattices. II. From the B-quadrilateral lattice to the self-adjoint schemes on the triangular and the honeycomb lattices*, Journal of Mathematical Physics, 48(11), 113506:1–113506:17.

[H8] P.M. Santini, M. Nieszporski, and A. Doliwa, 2008, *Integrable dynamics of Toda type on square and triangular lattices*, Physical Review E, 77(5), 056601:1–056601:12.

[H9] A. Doliwa and M. Nieszporski, 2009, *Darboux transformations for linear operators on two-dimensional regular lattices*, Journal of Physics A-Mathematical and Theoretical, 42(45), 454001:1–454001:27.

c) Discussion of the scientific aim of the above papers and the achieved results with a discussion of their possible applications.

1 Introduction

The notion of *continuity* is deeply rooted in the foundations of modern physics. The fundamental theories such as fluid mechanics, classical electrodynamics, quantum mechanics and general relativity are defined on differential manifolds and consequently the laws of these theories are formulated in the form of *differential equations*. The undoubted success of differential calculus in physical theories caused that the *difference equations* defined on discrete sets have been regarded as secondary with respect to their continuous counterparts for centuries. This situation changed with the development of numerical methods and e.g. due to attempts to quantize gravity. The need for the development of mathematical tools for *difference equations* appeared and this dissertation, which develops the theory of *discrete integrable systems*, belongs to the trend that responds to the demand.

In particular the difference geometry [1], the theory that tries to build a discrete world from scratch, however, without losing the correspondence with differential geometry (that is why this area is nowadays perversely referred to as the discrete differential geometry [2]) but also without blind copying the differential geometry, provides us with a guiding principle. Namely, first we confine ourselves to a certain class of surfaces having an additional feature, then we discretize the class of surfaces so that the resulting discrete lattices had the same feature and then we extend the results obtained in this way to more general objects that do not have the feature. In the case of the difference geometry *integrability* is the additional feature - first we choose a class of surfaces described by *integrable* nonlinear differential equations and discretize the class in such a way that the resulting class of lattices is described by *integrable* nonlinear difference equations.

By *integrability* of nonlinear equations, both difference ones and differential ones, we mean a series of interrelated properties including, first, the existence of Bäcklund transformations (allowing to construct from the known solutions of the equation its new solutions) and non-linear superposition principle (allowing to superpose these new solutions), second, the existence of the system of linear equations (Lax pair) which (what is very important from the point of view of this paper) is covariant under the so-called *Darboux type transformations* and for which compatibility conditions give the nonlinear equations in question, third, inverse scattering and algebro-geometric methods [3, 4, 5]. We will use the term *Darboux type transformations* in its broadest meaning, i.e. the binary Darboux transformation, which is often called the fundamental transformation, as well as its reductions, will be referred here to as Darboux type transformations. It will be important for us that many of the techniques for constructing solutions of nonlinear equations is based on these transformations, while we ignore the role of these transformations in the theory of linear equations.

Let us emphasize, the fundamental object in the continuous case is the class of the surfaces and not the differential equation describing the class. Focusing on the class of surfaces we unify differential equations that describe them. This statement, in the theory of integrable systems, we owe Antoni Sym [6]. The classical example are pseudo-spherical surfaces. The standard way to describe them is to give the angle $\phi(u, v)$ between asymptotic lines, the angle satisfies the sine-Gordon equation

$$\partial_u \partial_v \phi(u, v) = \sin \phi(u, v).$$

Another way of description is to give the normal vector $\vec{n}(u, v)$, which satisfies the following non-linear system (this system appears e.g. in theory of nonlinear σ-models or in theory of harmonic maps)

$$\partial_u \partial_v \vec{n}(u, v) = f(u, v)\vec{n}(u, v), \quad \vec{n}(u, v) \cdot \vec{n}(u, v) = 1$$

One can obtain the position vector $\vec{r}(u, v)$ of pseudo-spherical surfaces using the so-called Lelieuvre formulas

$$\partial_u \vec{r}(u, v) = \partial_u \vec{n}(u, v) \times \vec{n}(u, v), \quad \partial_v \vec{r}(u, v) = \vec{n}(u, v) \times \partial_v \vec{n}(u, v).$$

We will come back to Lelieuvre formulas in a moment.

We come to a burning problem of difference geometry. In the continuous case, we are able to change the parameterization of the surface surface $\tilde{u} = f(u, v)$, $\tilde{v} = g(u, v)$. In particular, the *linear differential equations of second order in two independent variables*, which can appear in the geometry of solitons e.g. as

- an equation of the Lax pair

- or a part of the non-linear system (see the above description of the pseudo-spherical surfaces by means of the normal vector)
- or equations of moving frame,

are usually reduced, by appropriate choice of the parameterization, to the canonical form

$$[\partial_x\partial_x + \epsilon\partial_y\partial_y + \alpha(x,y)\partial_x + \beta(x,y)\partial_y + \gamma(x,y)]\psi(x,y) = 0,$$

where $\epsilon = -1, 0, 1$. In the case of hyperbolic equations (i.e. when $\epsilon = -1$) we often take

$$[\partial_u\partial_v + a(u,v)\partial_u + b(u,v)\partial_v + c(u,v)]\psi(u,v) = 0 \tag{1}$$

as the canonical form. In difference geometry only some of specific parameterizations (coordinate nets) have been discretized so far, for example, the asymptotic coordinates. This restriction manifests on the level of equations in discretization of canonical form (1) only or, in other words, in the discretization of operator $\partial_u\partial_v + a(u,v)\partial_u + b(u,v)\partial_v + c(u,v)$. This is a weakness of differential geometry, for which we are trying to find a remedy here.

In the presented series of articles, we try to compensate the lack of a sufficiently general change of the independent variables in the discrete case, by considering more general discrete operators than those previously considered in the difference geometry. We present discretization of linear differential operators of second order in two independent variables. Classes of discrete operators we discuss here admit the Darboux type transformation in full analogy to their continuous counterparts and their less general discrete versions. In particular, we show the full (i.e. without specification to canonical form (1)) discretization of the linear operator of the second order in two independent variables, as well as the full discretization of formally self-adjoint linear operator of the second order in two independent variables. We show examples of the application of the obtained operators in the theory of integrable systems and difference geometry.

The necessity to consider the theory of discrete integrable systems on regular lattices other than $\mathbb{Z}^n$ was pointed out already in 1997 by S.P. Novikov [7, 8]. Considerations on the discretization of self-adjoint elliptic equations of second order in two independent variables so that the discrete operators were factorizable (and as a result so that the discrete operators admitted Laplace transformations) led to a 7-point self-adjoint operator given on the triangular lattice. Laplace transformations play an important role in the classification of differential and difference integrable equations and therefore the articles [7, 8] should be regarded as the beginning of applications of regular lattices other than $\mathbb{Z}^n$ in the theory of integrable systems. In the articles that the present dissertation consist of we went a step further. We showed e.g. that the 7-point operator also admits the Darboux type transformations (or more precisely their subclass referred to as Moutard type transformations) and as such can be considered as part of the Lax pairs for integrable systems. It is worth mentioning that opposite to the two-dimensional operators considered herein, for one-dimensional operators terms Darboux transformation and Laplace transformation can be used interchangeably.

Let us finally put the issue of discretization of differential equations in a broader perspective. Among the fundamental physics of nonlinear equations such as Einstein's equations, Navier-Stokes equations or Gross-Pitaevskii equations, their special cases such as respectively Ernst equation, the Kadomtsev-Petviashvili equation or nonlinear Schrödingera equation are integrable. An attempt can be made to discretize full equations (e.g. Einstein's ones or Navier-Stokes ones), starting from the results obtained for the integrable discretizations of integrable special cases of these equations. However, we have to learn first how discretize integrable equations so that they reproduce in the continuum limit not only an equation describing a geometrical situation but also the freedom that in the continuous case comes from the possibility of change of the independent variables. The first such example in the literature we recall in section 5 of this dissertation, where we discuss the results of article [H3] concerning the discretization Lelieuvre formulas of equi-affine geometry [9].

2 Darboux type transformations

The transformation for the operator $\frac{d^2}{dx^2} - u(x) - \lambda$, nowadays known as the Darboux transformation, was received by Darboux [10, 11] via separation of variables in the so-called Moutard transformation [12, 13]. We recall the Moutard transformation, since a large part of the presented dissertation is dedicated to this class of Darboux type transformations. Let ψ is the kernel of a Moutard operator

$$\mathcal{M} := \partial_x \partial_y - F(x, y) \tag{2}$$

and θ denotes particular non-zero element of the kernel $\mathcal{M}\theta = 0$ (particular solution of the Moutard equation), then set of functions $\tilde{\psi}$ given by solutions of the pair of linear differential equations

$$\partial_x(\theta\tilde{\psi}) = \theta\partial_x\psi - \psi\partial_x\theta, \quad \partial_y(\theta\tilde{\psi}) = -\theta\partial_y\psi + \psi\partial_y\theta,$$

belongs to the kernel of Moutard operator $\tilde{\mathcal{M}} := \partial_x\partial_y - \tilde{F}(x,y)$ where $\tilde{F} = \theta\partial_x\partial_y\frac{1}{\theta}$. Further studies on this type of transformations led to the construction of a transformation, not without reason referred to as *fundamental transformation*, for operators $\partial_u\partial_v + a(u,v)\partial_u + b(u,v)\partial_v + c(u,v)$ (or systems of such operators associated with a compatible system of equations of the form (1)) [14].

Let us present the essence of this transformation in a manner which can be used for both continuous and discrete case [H6,H9].

First, two operators of the second order $\mathcal{L}$ and L (differential ones or difference ones) we refer to as (gauge) equivalent iff there exist two operators of multiplication by a function (in the cases considered here, these functions are functions $\mathbb{R}^2 \to \mathbb{R}$ in the continuous case, while in the discrete case $\mathbb{R}^2$ should be replaced by an appropriate regular lattice) θ, ϕ such that $\mathcal{L} = \phi L\theta$. The point is that if we multiply L from the right side by an element of the kernel of the operator L, $L\theta = 0$ and on the left by an element of the kernel of the formally adjoint operator $L^\dagger$ to the operator L, $L^\dagger\phi = 0$ then equation $(\phi L\theta)\Psi = 0$ can be rewritten in the continuous case in the form $\partial_x P + \partial_y Q = 0$ and in the discrete case as $(T_m - 1)\mathcal{P} + (T_n - 1)\mathcal{Q} = 0$, which guarantees existence of the function $\tilde{\Psi}$ given in the continuous case by $P = \partial_y\tilde{\Psi}$, $Q = -\partial_x\tilde{\Psi}$, while in the discrete case via $\mathcal{Q} = (T_m - 1)\tilde{\Psi}$, $\mathcal{P} = -(T_n - 1)\tilde{\Psi}$. Given θ, ϕ (functional parameters of the transformation), then set of functions $\tilde{\Psi}$ obtained from set of functions Ψ belongs to the kernel of a second order linear operator, the so-called Darboux transform of operator L.

The key results of presented dissertation are contained in the papers [H1,H6] and in the review paper [H9], where we compare these results with the known results of the two-dimensional difference operators admitting Darboux-type transformations. Here we present only the main facts.

2.1 6-point scheme

In article [H6] we show, that the difference operator given on the six points of the triangular regular lattice

$$A_{m,n}T_mT_m + B_{m,n}T_nT_n + 2C_{m,n}T_mT_n + G_{m,n}T_m + H_{m,n}T_n + F_{m,n}, \quad (3)$$

where T_m and T_n denotes elementary shift operators on the triangular lattice, passing in the natural continuum limit into the most general linear operator in two independent variables

$$a(x,y)\partial_x\partial_x + b(x,y)\partial_y\partial_y + 2c(x,y)\partial_x\partial_y + g(x,y)\partial_x + h(x,y)\partial_y + f(x,y), \quad (4)$$

admits the Darboux type transformation having all the features of a fundamental transformation, see the Conclusion 2 of article [H6] (as well as Theorem 3.1 from [H9]).

It is important in the theory of integrable systems to question whether the class of operators can be restricted to a certain subclass of operators, which is preserved by a (reduced) Darboux type transformation. We deal with this issue in the article [H6] as well.

Constraints $A_{m,n} \equiv 0$, $B_{m,n} \equiv 0$ and $F_{m,n} \equiv 0$ are preserved by the transformation (Theorem 3.1 from [H9]). Setting $A_{m,n} \equiv 0$ and $B_{m,n} \equiv 0$, we get results for the two-dimensional case of the article [15]. Setting $A_{m,n} \equiv 0$, $B_{m,n} \equiv 0$ and $F_{m,n} \equiv 0$, leads to results for the two-dimensional case from the articles [16, 17]. These constraints do not change the nature of the transformation, the transformation still remains a fundamental transformation, its functional parameters are functions θ and ϕ. This procedure is referred in [H6] and [H9] to as a specification of the operator. Specifications $A_{m,n} \equiv 0$ and $B_{m,n} \equiv 0$ leads to a discretization of the hyperbolic operator appearing in the equation (1). In the continuous case the effect analogous to the specification $a(x,y) \equiv 0$ and $b(x,y) \equiv 0$ can be obtained by changing the parameterisation, i.e. a change of the independent variables. In the discrete case we need a drastic change of operator from given on the six points to the operator given on four points.

The class of operators can also by reduced by choice of a special gauge. Since this procedure is less important in the theory of integrable systems (in theory Darboux transformation we operate on equivalence classes of gauge equivalent operators) we will omit it here, we refer the interested reader to the articles [H6] and [H9]. Finally, we have at our disposal reductions, when the solutions ϕ of the equation formally adjoint to the equation can be expressed in terms of a solution θ of the equation by a difference substitution. We describe in Section 3.3 of [H9] two such cases, Goursat one and Moutard one, in the presence of constraints $A_{m,n} \equiv 0$ and $B_{m,n} \equiv 0$ and although both of these cases were known earlier (see [H14] and citations therein) investigation of the reduction of fundamental transformation to the Darboux type transformations for discrete Moutard equation and discrete Goursat equation is the original result of the paper [H9].

2.2 7-point self-adjoint scheme

In the article [H1] we show that formally self-adjoint difference operator from article [8]

$$a_{m,n}T_m + a_{m-1,n}T_m^{-1} + b_{m,n}T_n + b_{m,n-1}T_n^{-1} + c_{m+1,n}T_mT_n^{-1} + c_{m,n+1}T_m^{-1}T_n - f_{m,n}, \tag{5}$$

which is a discretization of the formally self-adjoint two-dimensional differential operator

$$\partial_x[A(x,y)\partial_x + C(x,y)\partial_y] + \partial_y[C(x,y)\partial_x + B(x,y)\partial_y] - F(x,y), \tag{6}$$

admits, in full analogy to its continuous counterpart, a kind Darboux type transformation: a Moutard type transformation. Moreover, both in continuous and in the discrete case, we are able to specify the operators, since both constraint $c_{m,n} \equiv 0$ and its continuous version $C(x,y) \equiv 0$ are preserved by mentioned transformations. In the discrete case specification leads to the operator

$$a_{m,n}T_m + a_{m-1,n}T_m^{-1} + b_{m,n}T_n + b_{m,n-1}T_n^{-1} - f_{m,n}, \tag{7}$$

which is a discrete counterpart of operator

$$\partial_x[A(x,y)\partial_x] + \partial_y[B(x,y)\partial_y] - F(x,y). \tag{8}$$

Again, in the continuous case the specification $C(x,y) \equiv 0$ can be obtained via change of independent variables.

These results are worth setting together with the known results on discretization of the two-dimensional formally self-adjoint differential operator. Namely the Moutard operator (2) has two discretizations: discrete Moutard operator [18, 19]

$$T_mT_n + 1 - f_{m,n}(T_m + T_n)$$

and adjoint discrete Moutard operator [H14]

$$T_mT_n + 1 - f_{m+1,n}T_m - f_{m,n+1}T_n.$$

It is easy now to understand our primary motivation. We wanted to find a full discretization of the equation (6) (but the one which admits Moutard type transformation) and not just the particular version (2). In particular, our main motivation was to investigate the discrete operators of elliptical nature i.e. schemes for which the discrete Dirichlet boundary value problem is the proper boundary value problem (see article [H3] of this dissertation).

The question remains whether there are connections between the discrete Moutard operator and operators (5) or operator (7)? We will present an affirmative answer to this question in the next section.

2.3 Q-difference operators

The paper [H4] shows that the results obtained in [H1] and [H6] can be rewritten in terms of q-differential operators. It is worth mentioning the MSc thesis written under my supervision [20], where fundamental transformation of q-differential operators was presented in a way closer to the original article by Jonas [14].

3 The sublattice approach

The particular form of discrete Moutard equation

$$[T_mT_n - 1 - \frac{p_m - q_n}{p_m + q_n}(T_m - T_n)]\psi(m,n) = 0$$

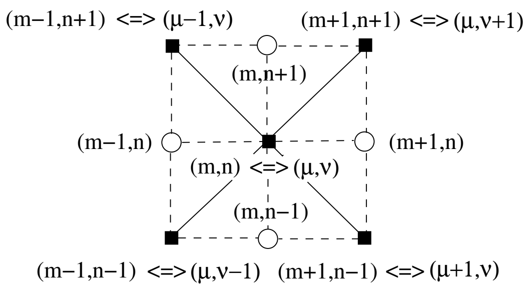
plays the crucial role in the theory of discrete holomorphic functions, see [21] and references therein (also introduction in [H5]), the equation is discrete counterpart of Cauchy-Riemann equations. To the best of my knowledge it is in the field of discrete holomorphic functions the sublattice approach was used for the first time, leading to discrete analog of Laplace equation. We give the general formulation of the approach on the second page of the paper [H5]. Here, we go straight to the examples, leading to the announced in the previous section links between discretizations of formally self-adjoint operator.

3.1 Discrete Moutard operator on $\mathbb{Z}^2$ lattice and the 5-point scheme

Proposition 1 and Proposition 2 from [H5] give a surprising characterization of discrete Moutard equations. Namely, having taken the linear equation $\alpha_{m,n}\psi_{m+1,n+1} + \beta_{m,n}\psi_{m+1,n} + \gamma_{m,n}\psi_{m,n+1} + \delta_{m,n}\psi_{m,n} = 0$ relating values of function ψ at four points of the square lattice $\mathbb{Z}^2$ and writing down this equation for four squares meeting at point (m, n)

$$\begin{aligned}
&\alpha_{m,n}\psi_{m+1,n+1} + \beta_{m,n}\psi_{m+1,n} + \gamma_{m,n}\psi_{m,n+1} + \delta_{m,n}\psi_{m,n} = 0,\\
&\alpha_{m-1,n}\psi_{m,n+1} + \beta_{m-1,n}\psi_{m,n} + \gamma_{m,n}\psi_{m-1,n+1} + \delta_{m-1,n}\psi_{m-1,n} = 0,\\
&\alpha_{m,n-1}\psi_{m+1,n} + \beta_{m,n-1}\psi_{m+1,n-1} + \gamma_{m,n-1}\psi_{m,n} + \delta_{m,n-1}\psi_{m,n-1} = 0,\\
&\alpha_{m-1,n-1}\psi_{m,n} + \beta_{m-1,n-1}\psi_{m,n-1} + \gamma_{m-1,n-1}\psi_{m-1,n} + \delta_{m-1,n-1}\psi_{m-1,n-1} = 0,
\end{aligned}$$

one can eliminate from them the four variables $\psi_{m-1,n}$, $\psi_{m+1,n}$, $\psi_{m,n-1}$ and $\psi_{m,n+1}$ iff the equation is gauge equivalent to a Moutard equation. The remaining five points satisfy the self-adjoint equation $L\psi = 0$, where L is operator (7) with suitably redefined shift operators on the lattice consisting of "black points" see Figure 1. This gives the relationship between the 5-point



Rysunek 1: From discrete Moutard equation to 5-point self-adjoint scheme.

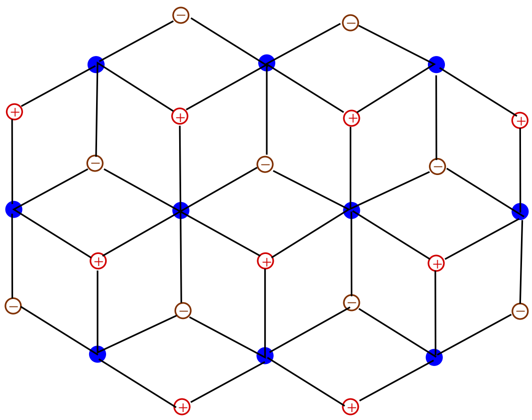
self-adjoint operator and discrete Moutard operator. More importantly, we showed that one can transfer integrable features from lattice to sublattice:

- from the Darboux type transformation and their superposition principle for discrete Moutard operator we derive Darboux type transformation and their superposition principle for self-adjoint 5-point scheme,
- from algebro-geometric solutions for the discrete Moutard equation we construct algebro-geometric solutions for self-adjoint 5-point equation.

3.2 Discrete Moutard operator on quasiregular rhombic tiling and the 7-point scheme

Analogously to the method we described in the previous subsection, one can get the relationship between the 7-point scheme and discrete Moutard operator (see paper [H7]). One has to replace lattice $\mathbb{Z}^2$ with quasiregular rhombic tiling (Figure 2).

On each rhombus of such tiling we prescribe a Moutard equation i.e. the sum (or difference) of the values of function ψ in opposite vertices of the rhombus has to be proportional to the sum (or difference) of the values of the function in the remaining two vertices of the rhombus. Taking six copies



Rysunek 2: Quasiregular rhombic tiling

of the equation for the six rhombi having common vertex of blue color (or grey in black and white printing) in Figure 2 and eliminating the values of function at the points marked with pluses and minuses we obtain in this way 7-point self-adjoint operator (5). It turns out that due to the sublattice approach one can deduce the Darboux type transformation for the 7-point self-adjoint operator (5) from the Moutard transformation for the discrete Moutard oparator, what we showed in Chapter IV A of article [H7].

3.3 Self-adjoint operator on the hexagonal lattice

Taking three copies of Moutard equation at three rhombi with a common vertex marked in Figure 2 with plus (or minus), we can eliminate values of function at the points marked in blue. This gives a pair of equations (and hence pair of operators) on a hexagonal lattice (which consists of the vertices marked with a plus and vertices marked with a minus in Figure 2). One equation of the pair relates values of function at point marked with plus and at three closest points marked with minus. In the second equation, conversely, the values of the function at the point marked with minus are related with values of the function in the closest three vertices denoted by a plus. As in the previous case, the sublattice approach allows us to derive Darboux type transformation for a pair of equations on a hexagonal lattice from Darboux type transformation for discrete Moutard equation (what we showed in section IV B of article [H7]).

3.4 A very short summary of the section 3

Figure 1 of review article [H9] (its full-scale version can be found in preprint of paper [H9] i.e. on Figure 1 from version [22]) collects our results and the results of other authors concerning the discretization of two-dimensional linear differential operators of second order.

4 Integrable generalization of Toda chains to square and triangular lattices

With a Darboux covariant linear operators (5) and (7) it is natural to ask about the non-linear equations associated with them, i.e. to treat these operators as part of the Lax pair (the linear problem) of a non-linear system. In articles [H2] and [H8] we showed that it leads to the genaralization of the so-called Toda chains [23] i.e. systems of equations of the form

$$\frac{d^2 Q_m(t)}{dt^2} = \Delta_m e^{\Delta_m Q_{m-1}(t)},$$

on function $Q_m(t)$ depending on continuous variable t and discrete variable m (where $\Delta_m = T_m - 1$). In case of operator (7) we get the following equations on $\mathbb{Z}^2$ lattice (see the article [H2] of this dissertation)

$$C_{m,n}\frac{d}{dt}\left(\frac{1}{C_{m,n}}\frac{dQ_{m,n}}{dt}\right) = \Delta_m\left(C_{m,n}C_{m-1,n}e^{\Delta_m Q_{m-1,n}}\right) + \Delta_n\left(C_{m,n}C_{m,n-1}e^{\Delta_n Q_{m,n-1}}\right),$$

$$\frac{C_{m+1,n+1}}{C_{m,n}} = e^{-\Delta_m\Delta_n Q_{m,n}},$$

In the case of the operator (5) considerations lead to the system of equations on a triangular lattice (see paper [H8] of this dissertation)

$$\frac{dF}{dt}+\frac{1}{\xi}\Delta_1\left(\xi\xi_{-1}A^2_{-1}\right)+\frac{1}{\eta}\Delta_2\left(\eta\eta_{-2}B^2_{-2}\right)+\frac{1}{\zeta}\Delta_3\left(\zeta\zeta_{-3}C^2_{-3}\right)=0,$$

$$\frac{1}{A}\frac{dA}{dt}+\frac{1}{2}\Delta_1(\xi F)-\frac{1}{2}\frac{BC_1}{A}(\eta+\zeta)_2+\frac{1}{2}\frac{B_{-3}C_{-3}}{A}(\eta+\zeta)_{-3}=0,$$

$$\frac{1}{B}\frac{dB}{dt}+\frac{1}{2}\Delta_2(\eta F)+\frac{1}{2}\frac{A_3C}{B}(\xi-\zeta)_3-\frac{1}{2}\frac{AC_1}{B}(\xi-\zeta)_1=0,$$

$$\frac{1}{C}\frac{dC}{dt}+\frac{1}{2}\Delta_3(\zeta F)+\frac{1}{2}\frac{B_{-1}A_{-1}}{C}(\xi+\eta)_{-1}-\frac{1}{2}\frac{A_3B}{C}(\xi+\eta)_2=0,$$

$$AB_1(\xi-\eta)_1=A_2B(\xi-\eta)_2,$$

$$AC(\xi+\zeta)=A_3C_1(\xi+\zeta)_2,$$

$$B_{-3}C_1(\eta-\zeta)_1=BC_{-3}(\eta-\zeta).$$

where to make formulas shorter we denoted the shift operators on a triangular lattice with subscripts, i.e. $T_if\equiv f_i$, $T_i^{-1}f\equiv f_{-i}$, (moreover $\Delta_i=T_i-1$) $i=1,2,3$. Please note that the shift operators on a triangular lattice are not independent, we have $T_1T_3=T_2$. Darboux type transformations for the operators considered in the previous sections allowed us to construct Bäcklund transformations for the above non-linear systems of equations. In the articles [H2] and [H8] we discuss the specific forms of the above non-linear equations as well as their reductions and their particular solutions.

5 Geometric aspects - Lelieuvre formulas

Revealing new integrable structures, in our case complicated (complex) structure of discrete equations that are discretizations of self-adjoint equations and that admit Moutard type transformations, usually finds application in discrete differential geometry. We focus here on Lelieuvre formulas, the classical issue of equi-affine geometry [9]. These formulas allows us to determine the surface from its (co)normal field and its affine fundametal form. With some not too restrictive assumptions (see conditions (31-33) of the article [H3]) one can show, that normal field to a surface in Euclidean space $\mathbb{E}^3$ (or more generally conormal field to a surface in three-dimensional equi-affine space

$e\mathbb{A}^3$) satisfies second order formally self-adjoint equation. When the equation is the Moutard equation ($M\psi = 0$ where M is (2)) formulas give the surface parameterized with the asymptotic lines. When the equation is of the form $L\psi = 0$, where L is operator (8), formulas give the surface parameterized with conjugate lines. Finally, generic equation $L\psi = 0$, where L is operator (6), reproduces after using Lelieuvre formulas generic parameterization of the surface.

In the discrete case, the discrete Moutard equation gives via Lelieuvre formulas discrete asymptotic lattices [24]. In article [H3] we showed, that Lelieuvre formulas for the equation associated with operator (7), i.e. discretization of (8), give quadrilateral lattices i.e. discrete analog of conjugate nets. Moreover, Lelieuvre formulas for the equation $L\psi = 0$ where L is operator (5) give generic (up to conditions (39-42) from article [H3]) two-dimensional lattices.

Thus we received another example, where understanding of the various aspects of integrable discretization allows us discretize classic constructions of differential geometry. However, for the first time in the discrete differential geometry we referred not only to the specific surface parameterization while discretizing but also to a generic parameterization of the surfaces.

6 The outlook

The theory of nonlinear discrete integrable equations reflecting not only the particular coordinate net on the soliton surfaces but also the freedom to change their parameterization is still in its infancy. Sublattice approach described in [H5] and [H7] was a promising candidate for analog of change of the independent variables for discrete equations. It seems, however, that it was the insufficient candidate. One of the possible directions of development of the theory was suggested by A. Doliwa in article [26], where sublattice approach was enriched with the possibility of making Laplace transformations and where it was shown that theory of quadrilateral latices [15] can be derived from the results of paper [16] on the Hirota-Miwa equation.

The most important, yet unsolved, problem is whether the results of article [H6] can be generalized to more dimensions in analogy to the papers [16] and [15]. In other words, whether there exists integrable discretization of the surfaces sustaining conjugate nets but without direct reference to the conjugate nets. It should be mentioned that the investigation of possibility of a generalization of integrable equations to *more dimensions* grew into a method of detection and study of discrete integrable equations [27, 25]. Although our papers in this area concerns only $\mathbb{Z}^n$ lattices [H17,H18,H19], our colleagues use the lattices other than $\mathbb{Z}^n$ lattices [28].

As we mentioned in the introduction, linear operators are often part of the system of nonlinear differential equations. Therefore a chance to discrtize the latter appears. Although this topic is not emphasized in a series of articles present dissertation consist of, we would like to present at least one example that *illustrates perspectives* of the methods used in the series of papers.

Namely, position vector $\vec{R}_{m,n}$ and its square of the length $|\vec{R}_{m,n}|^2$ of discrete isothermic surfaces satisfy a Moutard equation in the so-called affine gauge (see book [2])

$$\Delta_m\left(\Theta_{m,n}\Theta_{m,n+1}\Delta_n\begin{bmatrix}\vec{R}_{m,n}\\|\vec{R}_{m,n}|^2\end{bmatrix}\right)+\Delta_n\left(\Theta_{m,n}\Theta_{m+1,n}\Delta_m\begin{bmatrix}\vec{R}_{m,n}\\|\vec{R}_{m,n}|^2\end{bmatrix}\right)=0. \qquad (9)$$

In the continuum limit we obtain the equation for the position vector of discrete isothermic surface parameterized with curvature lines. Prescribing these equations on quasiregular rhombic tiling and applying sublattice approach we get

$$\Delta_m\left(A\Delta_{-m}\begin{bmatrix}\vec{R}\\|\vec{R}|^2\end{bmatrix}+C\Delta_{-n}\begin{bmatrix}\vec{R}\\|\vec{R}|^2\end{bmatrix}\right)+\Delta_n\left(B\Delta_{-n}\begin{bmatrix}\vec{R}\\|\vec{R}|^2\end{bmatrix}+C\Delta_{-m}\begin{bmatrix}\vec{R}\\|\vec{R}|^2\end{bmatrix}\right)=0,$$

where to make formulas shorter, we have omitted independent variables. The last system of equations in the continuum limit goes to the system of equations for the position vector of isothermic surfaces in the generic parameterization. We do not know geometric characterization of the discrete system of equations. We also failed to construct the Bäcklund transformation for the system of equations. On the other hand, the sublattice approach allows us to obtain solution of these equations from the solutions of system (9).

The list of open problems is much longer. A large part of monograph [2], which describes the results on integrable discretization of *specific* parameterizations on soliton surfaces, should be reviewed to check if it possible to generalize the results to generic parameterizations. Discretization of elliptic versions of nonlinear integrable differential equations such as the Ernst equation are not known yet.

Discrete differential geometry is still ahead of us.

5. Description of the remaining scientific achievments

Articles before PhD

[H10] M. Nieszporski and A. Sym, 2000, *Bäcklund transformations for hyperbolic surfaces in E^3 via Weingarten congruences*, Theoretical and Mathematical Physics, 122(1), 84–97.

[H11] M. Nieszporski, 2000, *The multicomponent Ernst equation and the Moutard transformation*, Physics Letters A, 272(1–2), 74–79.

[H12] A. Doliwa, M. Nieszporski, and P.M. Santini, 2001, *Asymptotic lattices and their integrable reductions: I. the Bianchi-Ernst and the Fubini-Ragazzi lattices*, Journal of Physics A-Mathematical and General, 34(48), 10423–10439.

[H13] M. Nieszporski, 2002, *On a discretization of asymptotic nets*, Journal of Geometry and Physics, 40(3–4), 259–276.

[H14] M. Nieszporski, 2002, *A Laplace ladder of discrete Laplace equations*, Theoretical and Mathematical Physics, 133(2), 1576–1584.

Articles after PhD

[H15] A. Doliwa, M. Nieszporski, and P.M. Santini, 2004, *Geometric discretization of the Bianchi system*, Journal of Geometry and Physics, 52(3), 217–240.

[H16] M. Nieszporski and Sym A, 2009, *Bianchi surfaces: integrability in an arbitrary parameterization*, Journal of Physics A-Mathematical and Theoretical, 42(40), 404014:1–404014:10.

[H17] P. Kassotakis and M. Nieszporski, 2011, *Families of integrable equations*, SIGMA, 7, 100:1–100:14.

[H18] P. Kassotakis and M. Nieszporski, 2012, *On non-multiaffine consistent-around-the-cube lattice equations*, Physics Letters A, 376(45), 3135–3140.

[H19] J. Atkinson and M. Nieszporski, 2013, *Multi-quadratic quad equations: Integrable cases from a factorized-discriminant hypothesis*, International Mathematics Research Notices,
e-pub: doi: 10.1093/imrn/rnt066.

Article [H10] is another example showing the unifying aspects of the theory of soliton surfaces. Results by Bianchi on a special subclass of Weingarten congruences i.e. so-called Bianchi congruences were used to construct the solutions of Ernst equation.

Article [H11] shows various integrable aspects of Weingarten congruences and is an itroduction to my PhD dissertation which allowed me to identify several integrable subclasses of Weingarten congruences.

Article [H13] discusses integrable aspects of discrete asymptotic nets and introduces the concept of discrete Weingarten congruence. As a result, it was possible to find a series of discrete integrable systems, in particular, the two discussed in [H12], namely the discrete Bianchi-Ernst system and discrete Fubini-Ragazzi system.

Article [H14] introduces the concept of a Laplace ladder of difference equations describing the quadrilateral lattices. The main findings of this article was the discovery of a discrete adjoint Moutard equation as well as the discovery of the discrete nonlinear Goursat equation.

In article [H15] we deal with the geometrical aspects of the discrete Bianchi-Ernst system (from article [H13]) and its Bäcklund transformations.

Article [H16] concerns integrable aspects of Bianchi surfaces given in generic parameterizations. It is a preparation to find integrable discretization of parameterizations of the surfaces other than asymptotic parameterization.

The articles [H17], [H18] i [H19] form a series of articles dedicated to systems of bond-vertex models given on lattice $\mathbb{Z}^n$, which admit the existence of a three-parameter family of scalar potentials (excluding the possibility of adding a constant to the potential). For specific values of parameters the potentials satisfy known consistent-around-the-cube equations [25] (which, due to consistency-around-the-cube are integrable). For other values of the parameters we obtain new multivalued recurences: correspondences consistent around the cube. By integrability of correspondence we understand the possibility to obtain their solutions from solutions of equations listed in the article [25] (due to Bäcklund transformations between these equations constructed by us).

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Self-presentation is a natural property of allpeople, without exception. In a natural way, it begins with the birth of a person's image. So, a person already from infancy collects a "puzzle" of his image, himself without realizing it.Â This natural process is called a natural distribution, as a result of which a person is determined in the system of social consciousness. But it is important to remember that within thisThe type of presentation process itself is not amenable to control and adjustment. A self-introduction explains who you are, what you do and what others need to know about you. You should provide a self-introduction any time you meet someone new and donâ€™t have a third party to introduce you. Offer a self-introduction when you are: Beginning an interview. Attending a hiring event. Networking with new connections. Giving a presentation.Â Self-introduction sample for a presentation. â€œGood afternoon. My name is Calob Cor and Iâ€™m the VP of Administration and Finance at Northern Investing. Self-presentational concerns can even underlie self-destructive behaviors, such as cigarette smoking and substance abuse (Sharp & Getz, 1996). The chapter begins by considering the nature of self-presentational behavior. Why do people engage in self-presentation, and when and how do they go about creating impressions of themselves in the minds of other people? In the second section of the.