Velocity Operators in Relativistic QED

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ABSTRACT. Velocity operators are considered in 3-space and 4-space formulations of relativistic quantum mechanics. The difficulties that occur in the former case result from a fundamental mismatch, originating in the role of time, between the kinematics and dynamics of relativistic QED.
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1. Introduction

The velocity operator of nonrelativistic quantum mechanics is simple and straightforward. Why, then, is the velocity operator of relativistic quantum mechanics so problematical? Bunge [1] has proposed that the root of the problem lies in the way in which the spatial position coordinates are used, and that a modified definition can remove the usual difficulties - though others then appear.

Here a different explanation is suggested: that the problem arises not from the spatial coordinates, but from the role that time is conventionally given. The foundation of the argument is that a quantum particle in Minkowski space should be described by a spacetime distribution function, rather than a spatial one, so that the time coordinate is not the evolution parameter.

This 4-space paradigm has been adopted by a number of researchers over several decades, with a variety of approaches and results [2, 3, 4]. Typically one constructs a manifestly invariant theory in which the space-time coordinates $X^\lambda = (x, y, z, ct)$ are all placed on an equal footing, i.e. both spatial position and time are regarded as observables. An invariant parameter τ, corresponding to the proper time of classical relativity, is used instead of t to describe the evolution of the wave function, which in the spin-$\frac{1}{2}$ theory proposed by the author [5, 6] is a 4-component Dirac spinor ψ. In this case, the fundamental observable derived from ψ is a real, Lorentz-invariant scalar function $\psi^\dagger(i\gamma^4)\psi \equiv F(\mathbf{X},\tau)$, where $\psi^\dagger$ is the usual conjugate transpose. (The conventions here are based on a Minkowski space with signature +++-, and a chiral representation of the spinors.)

F is interpreted as the expected density of charge in 4-space, and though it is not positive definite, it can be used to define expected values in the usual kind of way. (Thus F is a distribution function, not a probability density.) Although the theory is initially conceived as if for a single particle, it generally predicts the presence of both electrons and positrons, the latter normally at a low level indicating virtual pairs accompanying a real (i.e. permanently existing) electron.

The proper mass of a particle is now an observable, and is related to the parameter τ in a manner analogous to the usual energy-time relationship. When the proper mass is sharp, the wave equation reduces to the conventional one, with solutions that give the same energy levels as are usually found for a bound electron. Despite its unusual features, this version of Dirac's equation therefore reproduces the most important results of the standard theory. Its predictions for atomic charge distributions are in general not exactly the standard ones, but the differences are very small except for highly relativistic electrons.

2. The Velocity Operator of 4-space QED

The wave equation of the 4-space Dirac theory introduced above is

$$\hat{H}\psi \equiv ic\boldsymbol{\gamma}\cdot[\hat{\mathbf{P}} - \left(\frac{e}{c}\right)\mathbf{\Omega}]\psi \equiv ic\boldsymbol{\gamma}\cdot\hat{\mathbf{\Pi}}\psi = i\hbar\frac{\partial\psi}{\partial\tau}, \tag{1}$$

where $\boldsymbol{\gamma}$ represents the four Dirac matrices, and the other bold symbols are 4-vectors: $\hat{\mathbf{P}} = -i\hbar\boldsymbol{\partial}$ is the momentum operator, $\mathbf{\Omega}$ is the potential, and $\hat{\mathbf{\Pi}}$ is the kinetic-momentum operator. We also use a Lorentz-invariant inner product, which for the present purpose can be written as

$$\langle\alpha|\beta\rangle = \int \alpha^\dagger(i\gamma^4)\beta\, d^4X, \tag{2}$$

where (except in cases of special symmetry) the integration is over space-time, as indicated. Now suppose that a dynamical variable a, not explicitly τ-dependent, has the operator $\hat{a}$, i.e.

$$\langle a \rangle = \int \psi^\dagger (i\gamma^4) \hat{a} \psi \, d^4X, \tag{3}$$

(Our operators are 'G-Hermitian' - i.e. G and $G\hat{a}$ are Hermitian, though $\hat{a}$ need not be - where $G = i\gamma^4$ in this case. $\hat{H}$, as in (1), is an example.) Taking $d/d\tau$ of (3), and substituting for $\partial\psi/\partial\tau$ and $\partial\psi^\dagger/\partial\tau$ from (1), we introduce the Hamiltonian operator, and obtain a 4-space version of the usual result:

$$\frac{d}{d\tau}\langle a \rangle = \frac{i}{\hbar} \int \psi^\dagger (i\gamma^4) [\hat{H}, \hat{a}] \psi \, d^4X. \tag{4}$$

(The identity $\boldsymbol{\gamma}^\dagger \gamma^4 = -\gamma^4 \boldsymbol{\gamma}$ is needed here.) From

$$\langle \mathbf{X} \rangle = \int \psi^\dagger (i\gamma^4) \hat{\mathbf{X}} \psi \, d^4X, \tag{5}$$

where $\hat{\mathbf{X}} = \mathbf{X}$, the 4-space position vector, we define the 4-velocity operator $\hat{\mathbf{V}}$ by

$$\frac{d}{d\tau}\langle \mathbf{X} \rangle = \int \psi^\dagger (i\gamma^4) \hat{\mathbf{V}} \psi \, d^4X. \tag{6}$$

This therefore gives $\hat{\mathbf{V}} = (i/\hbar)[\hat{H}, \hat{\mathbf{X}}]$, and the result (again G-Hermitian) is

$$\hat{\mathbf{V}} = ic\boldsymbol{\gamma}. \tag{7}$$

As noted above, the proper mass m_0 is an observable: its operator is

$$\hat{m}_0 = \frac{-i\hbar}{c^2}\frac{\partial}{\partial\tau} = \frac{-i}{c}\boldsymbol{\gamma}\cdot\hat{\boldsymbol{\Pi}}, \tag{8}$$

and the classical definition $m_0 \mathbf{V} = \boldsymbol{\Pi}$ translates into

$$\frac{1}{2}(\hat{m}_0 \hat{\mathbf{V}} + \hat{\mathbf{V}} \hat{m}_0) = \hat{\boldsymbol{\Pi}} I \tag{9}$$

when we use the identity

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\eta^{\alpha\beta} I. \tag{10}$$

The conservation law in 4-space is

$$\frac{\partial F}{\partial \tau} + c\, div \mathbf{J} = \mathbf{0}, \tag{11}$$

where $F \equiv \psi^\dagger (i\gamma^4)\psi$ was introduced above and $\mathbf{J} \equiv -\psi^\dagger \gamma^4 \boldsymbol{\gamma} \psi$ is the particle 4-current - thus $J^4 = \psi^\dagger \psi$. In general, $F = F_1 - F_2$, where F_1 and F_2 are expected particle and antiparticle densities, and $\mathbf{J}$ is the sum of the particle and antiparticle currents. If these currents have a common 4-velocity $\mathbf{U}$, this is given by $c\mathbf{J} = (F_1 + F_2)\mathbf{U}$. On the other hand, the expected value obtained from the 4-velocity operator $ic\boldsymbol{\gamma}$ is

$$\langle \mathbf{V} \rangle = \int \psi^\dagger (i\gamma^4) ic\boldsymbol{\gamma}\psi\, d^4X = -c \int \psi^\dagger \gamma^4 \boldsymbol{\gamma} \psi\, d^4X = c \int \mathbf{J}\, d^4X, \tag{12}$$

though being a weighted average of 4-velocities, this is not itself a 4-velocity in general: it does not normally satisfy $\langle \mathbf{V} \rangle \cdot \langle \mathbf{V} \rangle = -c^2$.

As in conventional Dirac theory, the velocity operator - now in 4-space, not 3-space - can be used to give expected values. (Since we obtained it from an expected value, this is not surprising.) But when we turn to eigenvalues, the only result with any degree of plausibility is that $\hat{V}^4$ has eigenvalues $\pm c$. The spatial components $\hat{V}^k$ of the 4-velocity operator have eigenvalues $\pm ic$, and so the 4-velocity eigenstates cannot be taken seriously. As in the case of the usual 3-velocity operator $\hat{\mathbf{v}}$, no two components of $\hat{\mathbf{V}}$ allow simultaneous eigenstates.

3. Expected Values and Eigenvalues

The properties of $\hat{\mathbf{v}}$ tend to be mentioned only briefly in textbook treatments of Dirac's equation, and have long been regarded as problematical [1]. (Among the better-known texts, that of Sakurai [7] gives a fairly full account.) On the other hand, there are no difficulties in using the velocity operators of 3-space and 4-space to calculate the expected values $\langle \mathbf{v} \rangle$ and $\langle \mathbf{V} \rangle$. It is only when they are used to obtain eigenvalues that doubts arise. In other words, it is legitimate, in a relativistic setting, to write

$$\langle \mathbf{v} \rangle = \int \psi^\dagger (-c\boldsymbol{\alpha})\psi\, d^3x \tag{13}$$

where $\alpha^k = \gamma^4 \gamma^k$, or

$$\langle \mathbf{V} \rangle = \int \psi^\dagger (-c\gamma^4 \boldsymbol{\gamma})\psi\, d^4X, \tag{14}$$

but questionable whether $\hat{v}^i\psi = k^i\psi$ or $\hat{V}^\lambda\psi = K^\lambda\psi$ has any meaning.

While the integrals in (13) and (14) are vectors, i.e. have the appropriate transformation properties, the eigenvalues of the operators $\hat{\mathbf{v}} = -c\boldsymbol{\alpha}$ and $\hat{\mathbf{V}} = ic\boldsymbol{\gamma}$ are not. To put this remark in context, consider the non-relativistic quantum mechanics of a particle with spin. Suppose that the particle is in an eigenstate ψ of spin, with the spin axis given by the unit vector $\mathbf{n}$ and the eigenvalue k by $\mathbf{n}\cdot\boldsymbol{\sigma}\psi = k\psi$, where $\boldsymbol{\sigma}$ is the '3-vector' of Pauli spin matrices. Now rotate the axes so that $\mathbf{n}$ (which we are treating as a row vector) is transformed into $\bar{\mathbf{n}} = \mathbf{n}A^T$, and let a be an element of SU(2) that corresponds to the rotation matrix A : then ψ is transformed into $\bar{\psi} = a\psi$. Substituting for $\bar{\mathbf{n}}$ and $\bar{\psi}$, we need $\mathbf{n}A^T\cdot\boldsymbol{\sigma}a\psi = ka\psi$, and this is satisfied because of the condition

$$a^\dagger\boldsymbol{\sigma}a = A\boldsymbol{\sigma} \tag{15}$$

that relates A and a, defining a homomorphism between the groups SO(3) and SU(2) . (Note that $A^{-1} = A^T$ and $a^{-1} = a^\dagger$.)

Thus nonrelativistic quantum mechanics guarantees the invariance of spin eigenvalues under spatial rotations. However, the Dirac theory is more complicated: Lorentz invariance holds for the eigenvalues of 4-velocity, but only rotation invariance for those of 3-velocity and spin. A brief outline is as follows.

A relativistic generalization of the SU(2) result is obtained by using the four Dirac matrices γ^λ instead of the Pauli matrices. Specifically, the condition $\mathbf{N}\cdot\hat{\mathbf{V}}\psi = K\psi$, where $\mathbf{N}$ is a 4-vector and ψ is a bispinor, leaves the eigenvalue K invariant under Lorentz transformations of $\mathbf{N}$ and ψ. This should be expected, since the equation defining K is similar in form to the (conventional) free-field Dirac equation, in which the counterpart of K is the invariant proper mass, m_0. The invariance of K, like the Lorentz invariance of the Dirac equation, follows from a condition relating the matrices Λ of the Lorentz group, SO(3,1), to the matrices b of the homomorphic spinor-transforming group, SL(2,C). This condition, which extends (15) from purely spatial rotations (matrix A) to rotations and boosts (matrix Λ), is

$$b^\dagger\boldsymbol{\sigma}b = \Lambda\boldsymbol{\sigma}, \tag{16}$$

where $\boldsymbol{\sigma}$ now represents a '4-vector' of Pauli matrices, with $\sigma_4 = I$.

However, if we use $\gamma^4\boldsymbol{\gamma}$ rather than $\boldsymbol{\gamma}$, e.g. if we take the eigencondition to be $\mathbf{N}\cdot\gamma^4\boldsymbol{\gamma}\psi = K\psi$, then K is preserved by spatial rotations, but not by boosts. The same is true if we replace γ^4 with $\gamma^0 = -i\gamma^1\gamma^2\gamma^3$. The matrices γ^4 and γ^0 appear in the 4-vectors that may be formed from a bispinor ψ: all are generated by $\psi^\dagger\gamma^4\boldsymbol{\gamma}\psi$ and $\psi^\dagger\gamma^0\boldsymbol{\gamma}\psi$. We note especially that in standard Dirac theory the spin operators are $\Sigma^j = \gamma^0\gamma^j$, while the velocity operators are (to within a constant factor) $\alpha^j = \gamma^4\gamma^j$, where j = 1-3. (Here we choose $\mathbf{N}$ to be purely spatial in the laboratory frame.) Thus the spin and 3-velocity eigenvalues are rotation-invariant, but not Lorentz-invariant.

4. Discussion and Conclusions

If velocity eigenvalues are valid in non-relativistic quantum mechanics, why not also in a relativistic setting? Here the 4-space picture is helpful.

The fundamental principle distinguishing 4-space quantum mechanics from the usual formulation is that time is no longer a parameter, but an observable. It is conjugate to energy in exactly the same way as spatial position is to momentum; i.e. the distributions of t and E are related at any given τ by a Fourier transform of the appropriate wave functions. But in using the 3-velocity operator to obtain an eigenvalue, we implicitly require a precise value of t because of the derivative $d\langle\mathbf{v}\rangle/dt$. This is valid if t is the evolution parameter for functions on 3-space, but according to the 4-space view it means that the distribution of t is a δ-function, while that of E becomes infinitely dispersed, and so the probability that E exceeds any given value approaches unity. In the 3-space picture, where the proper mass m_0 is a fixed parameter, this implies that the speed $v \uparrow c$, in agreement with the eigenvalues of $\hat{\mathbf{v}}$. Thus the usual velocity operator gives the best answer to a question that should not be asked!

Relativistic dynamics offers another viewpoint, reinforcing the argument above. The classical relations $\mathbf{v} = \boldsymbol{\pi}/m$ and $\mathbf{V} = \boldsymbol{\Pi}/m_0$ give the 3-velocity $\mathbf{v}$ and 4-velocity $\mathbf{V}$ from the kinetic 3-momentum $\boldsymbol{\pi}$ and 4-momentum $\boldsymbol{\Pi}$. (Here m is the relativistic mass.) But in relativistic theory, because m depends on v, the 3-space relation $\mathbf{v} = \boldsymbol{\pi}/m$ cannot give $\hat{\mathbf{v}}$, though in a nonrelativistic setting it does of course give a valid velocity operator. And the 4-space relation $\mathbf{V} = \boldsymbol{\Pi}/m_0$ does not give $\hat{\mathbf{V}}$, because we can no longer assume that m_0 is only a parameter. We recall that in the 4-space picture m_0 is an observable, with operator $\hat{m}_0$ as in eq. (8), and so we should not expect to find an operator corresponding to $\boldsymbol{\Pi}/m_0$.

We can also ask how the velocity operator was arrived at, and whether there are any associated indications that its eigenvalues should be viewed with skepticism. In both 3-space and 4-space, the velocity operator is inferred from the rate of change of the expected value of position, and in both cases the initial result for $\langle \mathbf{v} \rangle$ or $\langle \mathbf{V} \rangle$ is entirely reasonable. But although, as we have noted, the transformation properties of the expected value are those of a vector in these two cases, the operators themselves (unlike $\hat{\boldsymbol{\pi}}$ and $\hat{\mathbf{\Pi}}$) are not vectors, and the eigenvalues that they produce are invariants - in one case under spatial rotations, and in the other under Lorentz transformations. This indicates that if the eigenvalues measure a physical property, then that property must be intrinsic to the particle. In other words, what we are getting is not a velocity (or a component of one), and it is here that we should see the warning flag.

What conclusions can we draw from all this? In the conventional formulation of relativistic QED, the time coordinate is given the relativistically unnatural role of evolution parameter. We thereby obtain a 3-velocity operator, though relativistic dynamics implies that we should not expect meaningful eigenvalues of velocity. The result of this discordance is an operator whose eigenvalues, $\pm c$, can be regarded as the best answer to an invalid question. If, on the other hand, we take a 4-space approach in which time is a coordinate, but is not also the evolution parameter, then the conflict between kinematics and dynamics is resolved: the 4-velocity operator gives expected values correctly, but its eigenvalues are clearly unphysical.

Finally, this brings us to the more general observation that an operator has two distinct roles: we use it to write an expected value (from which it may have originally been derived), and to obtain eigenvalues. In 4-space QED the distinction is underlined by the fact that operators can be G-Hermitian rather than Hermitian, and therefore will not necessarily give real eigenvalues, though expected values of real variables must be real. The 4-velocity operator has already provided an example: $\langle \mathbf{V} \rangle$ is real, and $\hat{V}^4$ gives real eigenvalues, but the $\hat{V}^j$ do not. In any case, if the expected values and the eigenvalues associated with a particular operator have different transformation properties, we ought to ask whether the eigenvalues are physically meaningful.

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Â©2019, Maha Sami. Nonrelativistic Operators for Relativistic Transition Rates. by Maha Sami. APPROVED BYÂ By using equivalent nonrelativistic operators obtained from the Foldy-Wouthuysen transformation and relativistically corrected SchroÂ¨dinger wave functions, we show that we obtain the same transition amplitude as in Dirac Theory up to order Î±2, where Î± is the ne structure constant. We show this for the one-electron case and provide a theoretical framework for the two-electron case. Relativistic Velocity Transformation. No two objects can have a relative velocity greater than c! But what if I observe a spacecraft traveling at 0.8c and it fires a projectile which it observes to be moving at 0.7c with respect to it!? Velocities must transform according to the Lorentz transformation, and that leads to a very non-intuitive result called Einstein velocity addition. Just taking the differentials of these quantities leads to the velocity transformation. Taking the differentials of the Lorentz transformation expressions for x' and t' above gives. Putting this in the not relativistic Rutherford (Mott) scattering is same as classical method.Â (Eq.3) where m is the particle's mass and v is the velocity. When the angle between the particle's momentums before (= P1) and after (= P2) scattering is Î¸, the change of the momentum is (Eq.4) changing cosÎ¸ into 1 - 2 sin2 (Î¸/2), (Eq.5). The change of the momentum is equal to the impulse (= Ft) the particle receives in the whole scattering process, as follows, (Eq.6) Here the force is Coloumb force between the two charges (e and Ze), and Ï†(t) is the angle between the directions of the impulse and r(t) (= position vector).Â Dirac equation of QED includes relativistic and spin effects. When we consider the relativistic effect, it is called Mott scattering. Unfortunately, from QED method we can not imagine the real world. 1, at relativistic velocities, the radiation is emitted into a narrow cone pointing in the direction of the electronâ€™s instantaneous motion. The total power P(Î»,t), in ergs secâˆ’1, radiated into all space at wavelength Î» by a single electron moving in a circular orbit of radius p with energy y in rest mass units is given by [4].Â One can easily confirm that the matrix representation of Ïƒ is a â€œloweringâ€ operator in the two-dimensional space with basis vectors. (98d). |2ã€‰â†’[10].